This document provides a summary of Recommendation 3 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Help students understand why procedures for computations with fractions make sense.

Building conceptual understanding is foundational for students' proficient use of computational procedures, but procedures with fractions are often taught without helping students understand how or why the procedures work. Teachers should help students build both procedural fluency and conceptual understanding by explaining to students how the computations procedures they are presenting transform fractions in meaningful ways. Using practices such as visual representations, estimation, and setting problems in real-world contexts help reinforce students' conceptual understanding.

How to carry out the recommendation

1. Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.

Instructional strategies from the examples

- Use visual representations and manipulatives to build insight into concepts underlying computational procedures and the reasons why these procedures work.
- Model the division of fractions with representations such as ribbons or a number line.
- Consider the advantages and disadvantages of different representations for teaching procedures for computing with fractions and whether a representation can be used with different types of fractions.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.1c, PS.2b, PS.2c, PS.4a, PS.4b, 6.NS.1, 6.RP.3b
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

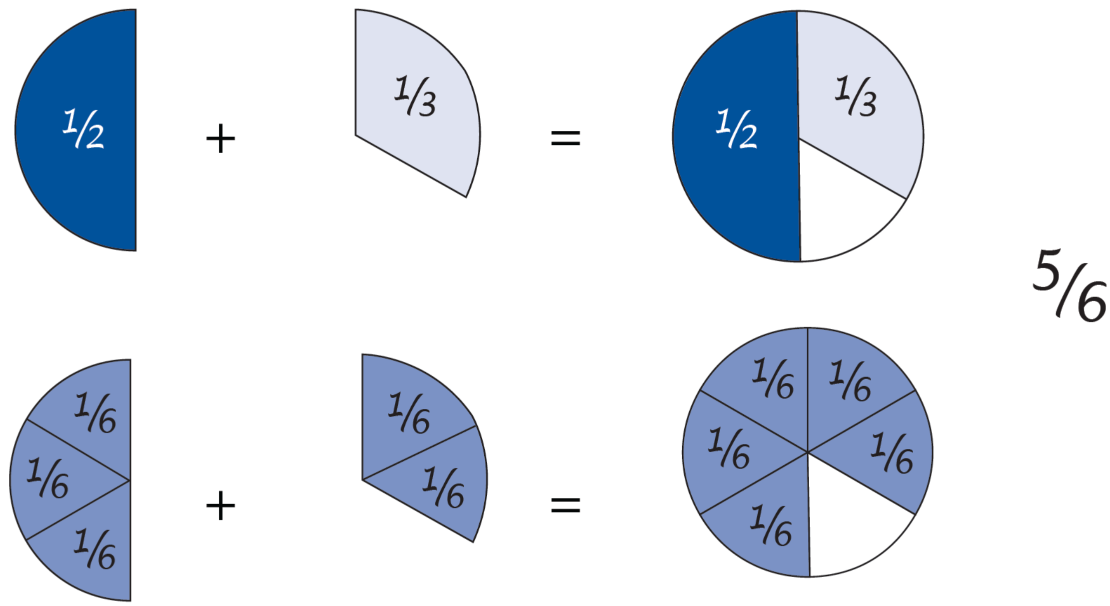
Teachers can use visual representations and manipulatives (e.g., number lines and area models) to help students understand basic concepts underlying conceptual procedures with fractions and the reasons why the procedures work. Here are some examples of how teachers can use representations of fractions in this way:

- **Find a common denominator when adding and subtracting fractions.** A common misunderstanding with students is that you can add fractions by adding the numerators and adding the denominators. Teachers can use a variety of representations to provide visual cues to help students see the need for common denominators. For example, if students are given two circle fractions representing $1/_2$ and $1/_3$, the teacher can show how converting both into sixths provides a common denominator that applies to both fractions, allowing the student to add the two fractions together (see Example 1 below). Teachers should build on this understanding by discussing with students why multiplying the denominators results in a common denominator that applies to both original fractions.
- **Redefine the unit when multiplying fractions.** Using concrete or pictorial representations can help students visualize that multiplying two fractions equates to finding a fraction of a fraction. Area models, for example, provide good examples, as shown in Example 2 below. The approach in Example 2 demonstrates how the initial unit is the full cake, treating the cake as the whole and dividing it into thirds, then moving to treat the portion shaded in the first step as the whole and dividing it into fourths.
- **Divide a number into fractional parts.** Although dividing fractions may look different on the surface, it is conceptually similar to dividing whole numbers. That is, students can think about how many times the divisor goes into the dividend. Teachers can use representations such as number lines and ribbons to model division of fractions. For example, in Example 3 below, ribbons are used to model $1/_2 \div 1/_4$, helping students think in terms of "How many $1/_4$s are there in $1/_2$?"

When selecting representation, teachers should consider the advantages and disadvantages of each to ensure that the representation adequately reflects the computational process students are expected to learn, allowing them to make connections between the representation and the computation. Teachers should also think about the facility of the representation in working with different types of fractions (e.g., proper fractions, mixed numbers, improper fractions, and negative fractions), as well as with other mathematical concepts (e.g., representing decimals with base-10 blocks, using 100 grids where one square is seen as $1/_{100}$ of the whole).

Example 1. Using fraction circles for addition

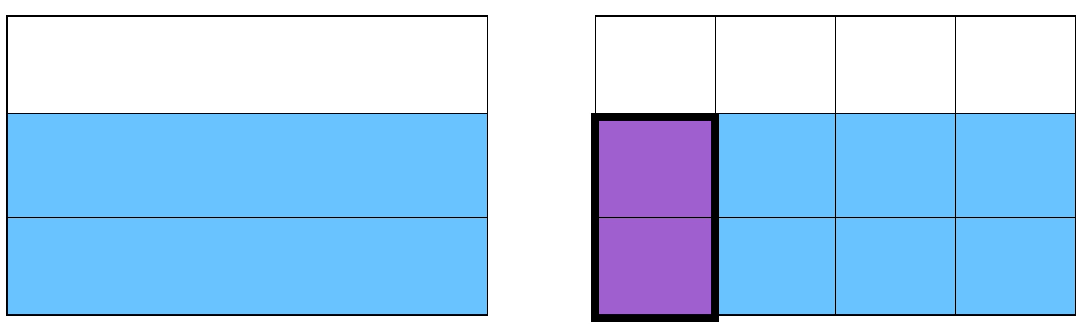
Adding $^1/_2 + {}^1/_3$ using fraction circles:



Note. Taken from Figure 6 on page 28 in the practice guide.

Example 2. Redefining the unit when multiplying fractions

Lori is icing a cake. She knows that 1 cup of icing will cover $^2/_3$ of a cake. How much cake can she cover with $^1/_4$ cup of icing?



$\frac{2}{3}$ of a cake

$\frac{1}{4}$ of $\frac{2}{3}$ of a cake

$\frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$ of a cake

Note. Taken from Figure 7 on page 29 in the practice guide.

Example 3. Using ribbons to model division with fractions

Students use ribbons to solve ${}^{1}/_{2} \div {}^{1}/_{4}$.

Step 1. Divide a ribbon into fourths.

${}^{1}/_{4}$	${}^{1}/_{4}$	${}^{1}/_{4}$	${}^{1}/_{4}$

Step 2. Divide a ribbon of the same length into halves.

${}^{1}/_{2}$	${}^{1}/_{2}$

Step 3. Compare the two ribbons to find out how many fourths of a ribbon are the same length as (that is, "can fit it into") one half.

${}^{1}/_{4}$	${}^{1}/_{4}$
⇩	⇩
${}^{1}/_{2}$	

Two-fourths are the same length as ("fit into") one half of the ribbon.

So ${}^{1}/_{2} \div {}^{1}/_{4} = 2$.

Note. Adapted from Figure 8 on page 30 in the practice guide.

2. Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.

Instructional strategies from the examples

- Provide opportunities for students to estimate the solutions to problems.
- Discuss whether and why students' solutions to specific problems are reasonable to help improve students' estimation skills.
- Explicitly teach estimation strategies.

South Carolina standards alignment

MATHEMATICS: PS.1d, PS.2a, PS.4c
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.5, INST.PS.1

In conjunction with teacher computation with fractions, providing students opportunities to estimate their solutions helps build their reasoning skills and focus on the meaning of the computational procedures. Teachers should ask students to provide an initial estimation and explain their thinking before having them actually compute the answer to help students judge the reasonableness of the answers they compute. Teachers can help improve both students' estimation skills and their judgment of the reasonableness of computed answers by having students discuss the strategies they used to determine their estimate and compare their initial estimates to the solutions they've computed.

Teachers should provide opportunities for students to estimate solutions for problems in which the solution cannot be easily computed. Additionally, explicitly teaching effective strategies for estimation can help maximize the value that estimation has on deepening students' understanding of fraction computation.

Example 4. Discussing estimation strategies and reasonableness of solution

Problem
${}^{1}/_{2} + {}^{1}/_{5}$

Solution
Student: I estimate that $^1/_2 + {}^1/_5$ is more than $^1/_2$ but less than $^3/_4$. **Teacher:** Why do you think that? **Student:** Well, I know that $^1/_2 + {}^1/_4 = {}^3/_4$. $^1/_5$ is less than $^1/_4$, so the answer for $^1/_2 + {}^1/_5$ will be less than $^3/_4$. **Teacher:** So, let's compute the sum and see. *Student computes the sum and arrives at a solution of* $^2/_7$ *because they incorrectly add the numerators and denominators.* **Teacher:** Do you think that $^2/_7$ is a reasonable solution for $^1/_2 + {}^1/_5$? **Student:** It can't be, because that's less than $^1/_2$, and I know the answer has to be bigger than $^1/_2$. *Teacher follows up with further discussion of the procedure the student used, helping them identify the error made and guiding them to understand the correct procedure and solution.*

Note. Adapted from example on page 31 in the practice guide.

Example 5. Strategies for estimating with fractions

Strengthening estimation skills can develop students' understanding of computational procedures.

Benchmarks. One way to estimate is through benchmarks—numbers that serve as reference points for estimating the value of a fraction. The numbers 0, $^1/_2$, and 1 are useful benchmarks because students generally feel comfortable with them. Students can consider whether a fraction is closest to 0, $^1/_2$, or 1. For example, when adding $^7/_8$ and $^3/_7$, students may reason that $^7/_8$ is close to 1, and $^3/_7$ is close to $^1/_2$, so the answer will be close to $1{}^1/_2$. Further, if dividing 5 by $^5/_6$, students might reason that $^5/_6$ is close to 1, and 5 divided by 1 is 5, so the solution must be a little more than 5.

Relative Size of Unit Fractions. A useful approach to estimating is for students to consider the size of unit fractions. To do this, students must first understand that the size of a fractional part decreases as the denominator increases. For example, to estimate the answer to ${}^{9}/_{10} + {}^{1}/_{8}$, beginning students can be encouraged to reason that ${}^{9}/_{10}$ is almost 1, that ${}^{1}/_{8}$ is close to ${}^{1}/_{10}$, and that therefore the answer will be about 1. More advanced students can be encouraged to reason that ${}^{9}/_{10}$ is only ${}^{1}/_{10}$ away from 1, that ${}^{1}/_{8}$ is slightly larger than ${}^{1}/_{10}$, and therefore the solution will be slightly more than 1. The principle can and should be generalized beyond unit fractions once it is understood in that context. Key dimensions for generalization include estimating results of operations involving non-unit fractions (e.g., ${}^{3}/_{4} \div {}^{2}/_{3}$), improper fractions (${}^{7}/_{3} \div {}^{3}/_{4}$), and decimals $(0.8 \div 0.33)$.

Placement of Decimal Point. A common error when multiplying decimals, such as 0.8×0.9 or 2.3×8.7, is to misplace the decimal. Encouraging students to estimate the answer first can reduce such confusion. For example, realizing that 0.8 and 0.9 are both less than 1 but fairly close to it can help students realize that answers such as 0.072 and 7.2 must be incorrect.

Note. Taken from Example 3 on page 31 in the practice guide.

3. Address common misconceptions regarding computational procedures with fractions.

Instructional strategies from the examples

- Identify students who are operating with misconceptions about fractions, discuss the misconceptions with them, and make clear both why the misconceptions lead to incorrect answers and why correct procedures lead to correct answers.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.2d
TEACHERS: INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.1, PLAN.SW.3, PLAN.Desc.1

Students' understanding of computational procedures with fractions can often be limited by misconceptions they carry with them about fractions. Once teachers have identified misconceptions in students, they can present these in discussions about why some procedures used by students result in correct answers, while others do not. Some common misconceptions are described here:

- **Believing that fractions' numerators and denominators can be treated as separate whole numbers.** A common mistake students make when adding or subtracting fractions is to try to add/subtract the numerators, then do the same with the denominators. This error is rooted in students applying their knowledge of whole numbers to fractions and not understanding that the denominator defines the size of the fractional part, while the numerator represents the number of parts of that size. Additionally, the fact that this approach works for the multiplication of fractions adds support for the misconception.

  To help students overcome this misconception, teachers should present meaningful problems to students, setting problems within real-world contexts that are relevant for their students. For example, page 32 in the practice guide presents the following: "If you have $^3/_4$ of an orange left and give $^1/_3$ of it to a friend, what fraction of the original orange do you have left?" In computing a solution, students should recognize that $\frac{3}{4} - \frac{1}{3} = \frac{2}{1}$ can't be correct because they can't come up with 2 oranges if they only started with $^3/_4$ of an orange in the first place. In using meaningful problems like this, teachers can help lay a foundation for students to think deeply about why treating the numerators and denominators as separate whole numbers is inappropriate, opening the door for discussion about appropriate procedures.

- **Failing to find a common denominator when adding or subtracting fractions with unlike denominators.** When adding or subtracting fractions with unlike denominators, a common mistake is that students just insert the larger denominator into the solution rather than convert both fractions to equivalent fractions with a common denominator. For example, when calculating $\frac{4}{5} + \frac{4}{10}$, students may find an incorrect solution of $\frac{8}{10}$. This misconception is rooted in students not understanding that different denominators represent different-sized unit fractions. A similar error of not converting the numerator of a fraction when changing the denominator is also rooted in this misconception (e.g., $\frac{2}{3} + \frac{2}{6}$ incorrectly becomes $\frac{2}{6} + \frac{2}{6}$). Teachers can use visual representations that show equivalent fractions (e.g., number lines, fraction strips) to help students see both the need for common denominators (i.e., equal sizes of unit-fraction parts) and appropriate changes to numerators.

- **Believing that only whole numbers need to be manipulated in computations with fractions greater than one.** Mixed numbers provide an additional challenge for students with misconceptions about fractions. Often, in addition/subtraction problems with mixed numbers, students may ignore the fractional part and work only with the whole numbers (e.g., $2\frac{2}{3} + 5\frac{1}{2} = 7$). Errors such as these may result from students ignoring the part of the problem they do not understand, misunderstanding the meaning of mixed numbers, or assuming that such problems simply have no solution.

  Related to this, students may think that the whole-number portion of a mixed number has the same denominator as the fraction in a problem. This misconception might lead students to incorrectly translate the problem $5 - \frac{2}{3}$ into $\frac{5}{3} - \frac{2}{3}$. This misconception might also lead students to want to add the whole number to the numerator in more complex problems. Page 32 in the practice guide presents the following example of this type of error: $3\frac{1}{3} \times \frac{6}{7} = \left(\frac{3}{3} + \frac{1}{3}\right) \times \frac{6}{7} = \frac{4}{3} \times \frac{6}{7} = \frac{24}{21}$.

  To help overcome this misconception, teachers should help students understand the relationship between mixed numbers and improper fractions. Understanding how to translate each into the other is crucial for working with these fractions.

- **Treating the denominator the same in fraction addition and multiplication problems.** Misconceptions about fractions also lead to errors like students mixing up procedures for adding and multiplying fractions. This often results in computational errors, such as students leaving the denominator unchanged in fraction multiplication problems involving fractions with common denominators (e.g., $\frac{2}{5} \times \frac{3}{5} = \frac{6}{5}$). Errors such as this might also result from the fact that students see more fraction addition problems than multiplication problems, leading them to generalize procedures from fraction addition incorrectly to multiplication.

Teachers can address this misconception by helping students focus on the conceptual basis for fraction multiplication. For example, the product $\frac{1}{2} \times \frac{1}{2}$ can be thought of as "half of a half" and students can visualize that half of one half would equal one fourth (i.e., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$). Additionally, teachers can help students think about the fact that multiplying by a fraction less than 1 results in a product that is smaller.

- **Failing to understand the invert-and-multiply procedure for solving fraction division problems.** When students lack a conceptual understanding, they often misapply procedures like "invert and multiply" (see Example 6 below). Errors like this often reflect a lack in conceptual understanding regarding why the invert-and-multiply procedure translates a multistep calculation into a more efficient way to calculate the correct quotient.

Example 6. Common student errors with the invert-and-multiply procedure

Problem
$\frac{2}{3} \div \frac{4}{5}$
Common Student Errors
Error 1—not inverting either fraction. Students may understand the procedure involves multiplying the two fractions but forget the "invert" part. $$\frac{2}{3} \div \frac{4}{5} = \frac{8}{15}$$ **Error 2—inverting the wrong fraction.** $$\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{4}{5} = \frac{12}{10}$$ **Error 3—inverting both fractions.** $$\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$$

Note. Taken from the examples on page 33 in the practice guide.

Teachers should ensure students understand the multistep calculation that is the basis of this procedure. The invert-and-multiply procedure is based in two mathematical concepts: (1) multiplying a number by its reciprocal results in a product of 1, and (2) dividing any number by 1 leaves the number unchanged. Teachers can show students how these connect to the invert-and-multiply procedure as follows for the problem $\frac{2}{3} \div \frac{4}{5}$:

- Multiplying both the dividend ($\frac{2}{3}$) and divisor ($\frac{4}{5}$) by the reciprocal of the divisor yields ($\frac{2}{3} \times \frac{5}{4}$) ÷ ($\frac{4}{5} \times \frac{5}{4}$).
- Multiplying the original divisor ($\frac{4}{5}$) by its reciprocal ($\frac{5}{4}$) produces a divisor of 1, which results in $\frac{2}{3} \times \frac{5}{4} \div 1$, which yields $\frac{2}{3} \times \frac{5}{4}$.
- Thus, the invert-and-multiply procedure, multiplying $\frac{2}{3} \times \frac{5}{4}$, provides the solution.

4. Present real-world contexts with plausible numbers for problems that involve computing with fractions.

Instructional strategies from the examples

- Use real-world contexts that provide meaning to the fractions involved in the problem.
- Tailor problems around details that are familiar and meaningful to the students; gather ideas from them.
- Make connections between a real-world problem and the fraction notation used to represent it.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, 6.RP.3, 7.RP.3, 8.EEI.5
TEACHERS: INST.MS.1, INST.PIC.2, INST.AM.6, PLAN.SW.3

Teachers should present students with problems that use plausible numbers in real-world contexts. The contexts should also provide meaning to both the fraction quantities involved and the computational procedures used to find a solution. Setting problems in real-world measurement contexts (e.g., using rulers, ribbons, or measuring tapes) and using food in the problem can help. When including food items, teachers should use both discrete (e.g., boxes of candy, apples) and continuous items (e.g., pizza, candy bars). A good source of ideas for relevant contexts are the students themselves, as they will help tailor problems around details familiar and meaningful to them (e.g., school events, field trips, activities in other subjects).

When providing real-world problems and contexts, teachers should try to help students connect the problem with the fraction notation used to represent it. At times, students can correctly solve a problem presented in a real-world context but struggle to solve the same problem when presented in formal notation. Teachers should continue to connect the notation back to the real-world story problem as students work through the solution.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Students make computational errors (e.g., adding fractions without finding a common denominator) when using certain pictorial and concrete object representations to solve problems that involve computation with fractions.	Teachers should carefully choose representations that map easily and most directly to the fraction computation they are teaching (e.g., demonstrating the need for similar units when adding fractions), as use of some representations can actually reinforce misconceptions. To reinforce the need for common unit fractions, teachers should consider using representations that hold units constant (e.g., measuring tapes).
When encouraged to estimate a solution, students still focus on solving the problem via a computational algorithm rather than estimating it.	Teachers should present estimation as a preliminary tool for helping anticipate the relative size and appropriateness of a solution, not a shortcut to an answer. In this, teachers should help students focus on the reasoning needed to estimate a solution. Posing problems that are not quickly solvable with mental computation (e.g., $\frac{5}{9}+\frac{3}{7}$ instead of $\frac{5}{8}+\frac{3}{8}$) will help avoid this roadblock.

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15