This document provides a summary of Recommendation 4 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Minimal**

Recommendation

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

"Thinking proportionally" means that students understand the multiplicative relationship between two quantities and is a critical skill students must develop in preparation for success in advanced mathematics. Ratios, rates, and proportions are contexts that require understanding of multiplicative relationships and lead into the cross-multiplication algorithm. Teachers should develop students' proportional reasoning skills before teaching the cross-multiplication algorithm using a progression of problems that help build their informal reasoning strategies. Teachers should also make sure to return to the informal reasoning strategies after teaching the cross-multiplication algorithm, demonstrating that the algorithm and the informal reasoning students use lead to the same answers. As a caution, studies involving many types of problem-solving have demonstrated that students often learn a strategy to solve a problem in one context but fail to successfully generalize to other contexts.

How to carry out the recommendation

1. Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples

- Use a progression of problems that builds on students' developing strategies for proportional reasoning.
- Encourage students to apply their own strategies, discuss the various strategies' strengths and weaknesses, and help them understand why a problem's solution is correct.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, 6.RP.1, 6.RP.2, 7.RP.2a
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

Teachers should provide opportunities for students to solve ratio, rate, and proportion problems, building students' strategies for proportional reasoning, before teaching the cross-multiplication algorithm. While encouraging students to apply their own strategies, teachers should also introduce ways to solve these problems if students struggle with generating their own strategies. In this, teachers should discuss the various strengths and weaknesses of strategies and help students understand why the resulting solution is correct.

A possible progression teachers might use to guide students from informal proportional reasoning strategies to the cross-multiplication algorithm might look like this (see Example 1):

- **Buildup Strategy.** Initially pose story problems that allow students to use a buildup strategy where they repeatedly add the numbers within one ratio to solve a problem. In these problems, teachers should ensure the numbers in the ratios are integrally related so that one can be generated by repeatedly adding the numbers in the other (e.g., 2:3 and 10:15). These should begin with problems involving smaller numbers to allow students to build their understanding, then progress to problems with larger numbers to demonstrate how time-consuming it can be to repeatedly add to these large numbers. This will help students recognize the value of using multiplication and division.

- **Unit Ratio Strategy.** Next, teachers can present problems that cannot be easily solved through repeated addition or through multiplying/dividing by a single integer (e.g., $\frac{x}{6} = \frac{3}{9}$). Solving these types of problems involves reducing the known ratio to a form with a numerator of 1, then determining the multiplicative relationship between the denominator in the new unit ratio and that in the ratio with the unknown value. This relationship can then be used to solve for the unknown value. This strategy can also be used in problems where the solution is not a whole number and bridge into helping students see the value in the cross-multiplication algorithm.
- **Cross-Multiplication.** Building on students' understanding of the unit ratio strategy, teachers can present problems that do not involve integral relations or that use ratios that cannot be easily reduced to unit fractions. This will help students see the advantages of using a strategy that can help solve problems, regardless of the numbers involved. Teachers should continue to help students make connections by having them return to previous strategies to see that cross-multiplication results in the same answer and discussing why this is the case.

Teachers should continue to present problems that can be solved easily through informal reasoning and mental mathematics, as well as those more easily solved with cross-multiplication. In this, teachers can discuss with students how to anticipate which approach might be easiest for a given problem.

Example 1. Problems encouraging specific strategies

Buildup Strategy
Sample problem. If Steve can purchase 3 baseball cards for \$2, how many baseball cards can he purchase with \$10? **Solution approach.** Students can build up to the unknown quantity by starting with 3 cards for \$2, and repeatedly adding 3 more cards and \$2, thus obtaining 6 cards for \$4, 9 cards for \$6, 12 cards for \$8, and, finally, 15 cards for \$10.

Unit Ratio Strategy
Sample problem. Yukari bought 6 balloons for $24. How much will it cost to buy 5 balloons? **Solution approach.** Students might figure out that if 6 balloons costs $24, then 1 balloon costs $4. This strategy can later be generalized to one in which eliminating all common factors from the numerator and denominator of the known fraction does not result in a unit fraction (e.g., a problem such as $\frac{6}{15} = \frac{x}{10}$, in which reducing $\frac{6}{15}$ results in $\frac{2}{5}$).
Cross-Multiplication
Sample problem. Luis usually walks the 1.5 miles to his school in 25 minutes. However, one of the streets on his usual path is being repaired today, so he needs to take a 1.7-mile route. If he walks at his usual speed, how much time will it take him to get to his school? **Solution approach.** This problem can be solved in two stages. First, because Luis is walking at his "usual speed," students know that $\frac{1.5}{25} = \frac{1.7}{x}$. Then, the equation may be most easily solved using cross-multiplication. Multiplying 25 and 1.7 and dividing the product by 1.5 yields the answer of $28\frac{1}{3}$ minutes, or 28 minutes and 20 seconds. It would take Luis 28 minutes and 20 seconds to reach school using the route he took today.

Note. Taken from Example 4 on page 38 of the practice guide.

Example 2. Why cross-multiplication works

Teachers can explain why the cross-multiplication procedure works by starting with two equal fractions, such as $\frac{4}{6} = \frac{6}{9}$. The goal is to show that when two equal fractions are converted into fractions with the same denominator, their numerators also are equivalent. The following steps help demonstrate why the procedure works.

Step 1. Start with two equal fractions, for example: $\frac{4}{6} = \frac{6}{9}$.

Step 2. Find a common denominator using each of the two denominators.

a. First, multiply $\frac{4}{6}$ by $\frac{9}{9}$, which is the same as multiplying $\frac{4}{6}$ by 1.

b. Next, multiply $\frac{6}{9}$ by $\frac{6}{6}$, which is the same as multiplying $\frac{6}{9}$ by 1.

Step 3. Calculate the result: $\frac{(4\times9)}{(6\times9)} = \frac{(6\times6)}{(9\times6)}$

Step 4. Check that the denominators are equal. If two equal fractions have the same denominator, then the numerators of the two equal fractions must be equal as well, so $4 \times 9 = 6 \times 6$.

Note that in this problem, $4 \times 9 = 6 \times 6$ is an instance of $(a \times d = b \times c)$.

As a result, students can see that the original proportion, $\frac{4}{6} = \frac{6}{9}$, can be solved using cross-multiplication, $4 \times 9 = 6 \times 6$, as a procedure to create equivalent ratios efficiently.

Note. Taken from Example 5 on page 39 of the practice guide.

2. Encourage students to use visual representations to solve ratio, rate, and proportion problems.

Instructional strategies from the examples

- Select representations that are likely to elicit insight into a particular aspect of ratio, rate, and proportion concepts.
- Encourage students to create their own representations.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1c, PS.2b, PS.4a, PS.5a, 6.RP.2a, 6.RP.3b, 7.RP.2, 8.EEI.5c
TEACHERS: INST.PIC.2, INST.TCK.2

Teachers should encourage students to use visual representations when solving ratio, rate, and proportion problems and select representations that highlight specific concepts within the problem. For example, teachers can use a ratio table to represent the relations in a proportion problem and provide specific reference to highlight how multiplication leads to the same solution as the buildup strategy (see Example 3). In addition, teachers can use ratio tables to help students explore different aspects of proportional relationship, such as multiplicative relationships within and between ratios (see Example 3).

Teachers should encourage students to develop their own representations and not always provide them. With ratios, rates, and proportions, students initially tend to use tabular or other systematic record-keeping formats. Through formal instruction and exposure, teachers can introduce other representations and encourage students to use these in future problems.

Example 3. Ratio table for a proportion problem

Problem
How many cups of flour are needed for 32 people when a recipe calls for 1 cup of flour to serve 8 people?

Solution

Students can use a ratio table to repeatedly add 1 cup of flour per 8 people to find the correct amount for 32 people.

Cups of Flour	1	2	3	4
Number of People Served	8	16	24	32

Students can also use the ratio table to see that multiplying by the ratio $\frac{4}{4}$ (i.e., four times the recipe) provides the amount of flour needed for 32 people. Alternatively, the number of people served is always 8 times the number of cups of flour needed; thus, the ratio between them is 1:8.

Note. Adapted from Example 9 on page 39 and Example 10 on page 40 of the practice guide.

3. Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples

- Focus instruction on the meaningful features of different problem types so students can transfer their learning to new situations.
- Help students identify key information needed to solve a problem and how to use diagrams or pictures to depict that information.
- Encourage students to use different diagrams and strategies to arrive at solutions.
- Provide opportunities for students to compare and discuss their diagrams and strategies.
- Provide real-life contexts in problems.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.1d, PS.2c, PS.2d, PS.3b, PS.3d, PS.5a, PS.7b, 7.RP.2, 7.RP.2d
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.9, INST.TCK.2, PLAN.SW.1, PLAN.Desc.1

Teachers should focus instruction on meaningful features of different ratio, rate, and proportion problems to help students identify problems with common underlying structures. The goal is for students to transfer their learnings to new situations and contexts. For example, a recipe problem might call for 3 eggs to make 20 cupcakes and ask students to find the number of eggs for 80 cupcakes, while in another problem, building 3 doghouses requires 42 boards, and students need to determine how many boards are needed for 9 doghouses.

To develop students' ability to generalize to other contexts, teachers should first help them identify key information they will need to solve the problem. Then, teachers can teach students how to use diagrams to represent that information, focusing the diagrams on not only depicting the information but also the relationships between different quantities in the problem. Teachers should also provide opportunities for students to compare and discuss their various diagrams and strategies.

Finally, teachers should set ratio, rate, and proportion problems within real-life contexts, such as unit price, scaling, recipes, mixture, and time/speed/distance.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
Many students misapply the cross-multiplication strategy.	Teachers can carefully present several examples of why cross-multiplication works, following the process in Example 2, to help students understand the logic behind the procedure. This will also help students see why the correct form of a ratio problem is necessary for the procedure to work.
Some students rely nearly exclusively on the cross-multiplication strategy for solving ratio, rate, and proportion problems, failing to recognize that there often are more efficient ways to solve these problems.	Teachers should encourage a variety of strategies for solving ratio, rate, and proportion problems. Presenting problems that are easier to solve with strategies other than cross-multiplication encourages students to use prior knowledge. For example, to find the solution for $\frac{5}{15} = \frac{6}{x}$, students can see that 15 is a multiple of 5 (i.e., $5 \times 3 = 15$), so *x* will be the same multiple of 6 (i.e., $x = 6 \times 3$). Requiring students to solve problems mentally can also help them see and use other strategies, as well as build number sense.
Students do not generalize strategies across different ratio, rate, and proportion contexts.	When presenting problems set in different contexts, teachers should also make sure to link these new problems with problems students have previously solved. Teachers can also encourage students to judge whether the same solution strategy can be used for different types of problems (e.g., recipe and mixture problems can be organized the same way so solutions can be compared side by side).

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15