This document provides a summary of Recommendation 5 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Help students recognize and articulate math concepts and notation.

Students who have a solid understanding of math concepts and notation are better able to identify the math content of a problem, adapt their current knowledge to new problems, and consider multiple approaches to solving problems. Teachers can convey math concepts and notation by using problem-solving activities, by asking students to use mathematically valid explanations to explain how a worked problem was solved, and by introducing students to algebraic notation in a systematic manner.

How to carry out the recommendation

1. Describe relevant math concepts and notation, and relate them to the problem-solving activity.

Instructional strategies from the examples

- Watch and listen for opportunities to call attention to mathematical concepts and notation students use as they solve problems.
- Draw attention to mathematical ideas and concepts by directly instructing students in them before engaging the students in problem-solving.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, PS.2d, PS.4b
TEACHERS: INST.PIC.2, INST.TCK.2

Teachers can help students learn to connect their existing math intuition to formal concepts and notation. Teachers can do this is by watching and listening while students solve problems and pointing out formal concepts and notations as they are used. If students use informal language or notation for a concept, teachers can translate it into formal mathematical language and explain that there are multiple ways to communicate the same idea, moving students toward more formal mathematical approaches. Occasionally, teachers may need to explicitly instruct students in formal math concepts and notation before proceeding to problem-solving.

Example of students' intuitive understanding of formal math concepts

Problem
Is the sum of two consecutive numbers always odd?

Solution

STUDENT: Yes.

TEACHER: How do you know?

STUDENT: Well, suppose you take a number, like 5. The next number is 6.

For 5, I can write five lines, like this:

| | | | |

For 6, I can write five lines and one more line next to it, like this:

| | | | | |

Then, I can count all of them, and I get 11 lines. See? It's an odd number.

TEACHER: When you say, "It's an odd number," you mean the sum of the two consecutive numbers is odd. So can you do that with any whole number, like n? What would the next number be?

STUDENT: It would be $n + 1$.

TEACHER: So can you line them up like you did for 5 and 6?

STUDENT: You mean, like this?

n

$\underline{n + 1}$

TEACHER: Right. So what does that tell you about the sum of n and $n + 1$?

STUDENT: It's 2 ns and 1, so it's odd.

TEACHER: Very good. The sum, which is $n + n + 1 = 2n + 1$, is always going to be odd.

Note. Taken from example 18 on page 41 of the practice guide.

2. Ask students to explain each step used to solve a problem in a worked example.

Instructional strategies from the examples

- Provide students with opportunities to explain the process used to solve a problem in a worked example and to explain why the steps worked.
- Use small-group activities to encourage students to discuss the process used in a worked example and the reasoning for each step.
- Use probing questions to help students articulate mathematically valid explanations.

South Carolina standards alignment

MATHEMATICS: PS.2d, PS.3a, PS.6c, PS.7c
TEACHERS: INST.AM.4, INST.TCK.2, PLAN.Desc.1

Teachers should give students an opportunity to explain what problem-solving process they used in a worked example as well as why that process worked. Students can discuss the problem-solving process in small groups or restate one another's explanations in pairs. If students struggle to come up with mathematically valid explanations, teachers should ask questions to guide students to more valid explanations. Teachers also can provide examples of valid explanations or reword students' explanations.

An abridged example of student explanations

Problem
Are ${}^{2}/_{3}$ and ${}^{8}/_{12}$ equivalent fractions?
An explanation that is not mathematically valid
Student: To find an equivalent fraction, whatever we do to the top of ${}^{2}/_{3}$ we must also do to the bottom. **Teacher:** What do you mean? **Student:** It just works when you multiply it.

Teacher: What happens when you multiply in this step?

Student: The fraction . . . stays the same.

Teacher: That's right. When you multiply a numerator and denominator by the same number, you get an equivalent fraction. Why is that?

Student: Before there were 3 parts, but we made 4 times as many parts, so now there are 12 parts.

Teacher: Right, you had 2 parts of a whole of 3. Multiplying both by 4 gives you 8 parts of a whole of 12. That is the same part-whole relationship—the same fraction, as you said. Here's another way to look at it: When you multiply the fraction by $^4/_4$, you are multiplying it by a fraction equivalent to 1; this is the identity property of multiplication, and it means when you multiply anything by 1, the number stays the same.

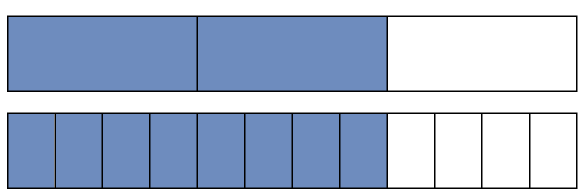
A correct description, but still not a complete explanation

Student: Whatever we multiply the top of $^2/_3$ by we must also multiply the bottom by.

A mathematically valid explanation

Student: You can get an equivalent fraction by multiplying the numerator and denominator of $^2/_3$ by the same number. If we multiply the numerator and denominator by 4, we get $^8/_{12}$.

If I divide each of the third pieces in the first fraction strip into 4 equal parts, then that makes 4 times as many parts that are shaded and 4 times as many parts in all. The 2 shaded parts become $2 \times 4 = 8$ smaller parts, and the 3 total parts become $3 \times 4 = 12$ total smaller parts. So the shaded amount is $^2/_3$ of the strip, but it is also $^8/_{12}$ of the strip:



Note. Adapted from Example 19 on page 42 of the practice guide.

3. Help students make sense of algebraic notation.

Instructional strategies from the examples

- Introduce symbolic notation early and at a moderate pace, allowing students enough time to become familiar and comfortable with it.
- Ask students to explain each component of an algebraic equation by having them link the equation back to the problem they are solving.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2d
TEACHERS: INST.TCK.2, PLAN.SW.3

To allow students time to become comfortable with the symbolic notation used in algebra, teachers should introduce symbolic notation early and at a rate that is not too slow or too fast for learners. One way of accomplishing this is to provide students with arithmetic problems and then support them in translating the problems into algebraic notation. This approach will help students connect their existing arithmetic knowledge with new algebraic knowledge.

Example of linking components of an equation to a problem

Problem
Joseph earned money for selling 7 CDs and his old headphones. He sold the headphones for $10. He made $40.31. How much did he sell each CD for?
Solution
The teacher writes this equation: $$10 + 7x = 40.31$$ **TEACHER:** If x represents the number of dollars he sold the CD for, what does the $7x$ represent in the problem? What does the 10 represent? What does the 40.31 represent? What does the $10 + 7x$ represent?

Note. Adapted from Example 21 on page 43 of the practice guide.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
Students' explanations are too short and lack clarity and detail. It is difficult for teachers to identify which mathematical concepts they are using.	Teachers can determine what concepts students are likely to use by solving problems before using them in lessons. Teachers can ask students questions about how a problem was solved and how they thought about the problem. They can also ask students to make a sheet of mathematical rules and use those rules when explaining how they approached a specific problem. The sheet should be brief and include only a few key rules.
Students may be confused by mathematical notations used in algebraic equations.	Teachers should use, and encourage students to use, arbitrary variables in order to help students understand that variables play an abstract role in algebraic equations. For example, use x or y to represent an unknown quantity. Students may confuse variables if they seem related to items in a problem (for example, if a represents apples).

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16