WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
Improving Mathematical Problem Solving in Grades 4 Through 8				
Recommendation 1: Prepare problems and use them in whole-class instruction.	Minimal	1. Include both routine and non-routine problems in problem-solving activities. (***PR.1.1***)	PS.1c	INST.MS.1, INST.AM.4, INST.TCK.2, PLAN.SW.1
		2. Ensure that students will understand the problem by addressing issues students might encounter with the problem's context or language. (***PR.1.2***)	PS.2d	INST.PIC.2, INST.TCK.2, PLAN.Desc.1
		3. Consider students' knowledge of mathematical content when planning lessons. (***PR.1.3***)	PS.1a	INST.AM.6, PLAN.SW.3, PLAN.Desc.1
Recommendation 2: Assist students in monitoring and reflecting on the problem-solving process.	Strong	1. Provide students with a list of prompts to help them monitor and reflect during the problem-solving process. (***PR.2.1***)	PS.1b, PS.1c	INST.AM.5, INST.TCK.2, INST.PS.1, PLAN.SW.1
		2. Model how to monitor and reflect on the problem-solving process. (***PR.2.2***)	PS.1c, PS.3d	INST.PIC.3, INST.AM.5, INST.PS.1, PLAN.SW.1
		3. Use student thinking about a problem to develop students' ability to monitor and reflect. (***PR.2.3***)	PS.1a, PS.1c, PS.3a	INST.MS.1, INST.MS.2, INST.AM.5, INST.PS.1, PLAN.SW.1, PLAN.Desc.1
Recommendation 3: Teach students how to use visual representations.	Strong	1. Select visual representations that are appropriate for students and the problems they are solving. (***PR.3.1***)	PS.2b	INST.PIC.2, INST.AM.6, INST.TCK.2
		2. Use think-alouds and discussions to teach students how to represent problems visually. (***PR.3.2***)	PS.1c, PS.2b, PS.4a, PS.7c	INST.MS.2, INST.PIC.2, INST.PIC.3, INST.TCK.2, INST.PS.1
		3. Show students how to convert the visually represented information into mathematical notation. (***PR.3.3***)	PS.1c, PS.2a, PS.2b, PS.4a, PS.4b	INST.PIC.2, INST.TCK.2
Recommendation 4: Expose students to multiple problem-solving strategies.	Moderate	1. Provide instruction in multiple strategies. (***PR.4.1***)	PS.1b, PS.1c	INST.PIC.2, INST.AM.9, INST.TCK.2, INST.PS.1
		2. Provide opportunities for students to compare multiple strategies in worked examples. (***PR.4.2***)	PS.1b, PS.1c, PS.3b, PS.3d, PS.7b, PS.7c	INST.MS.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.1

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
		3. Ask students to generate and share multiple strategies for solving a problem. (***PR.4.3***)	PS.1b, PS.1c, PS.3a, PS.5a, PS.7b, PS.7c	INST.MS.1, INST.MS.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1
Recommendation 5: Help students recognize and articulate mathematical concepts and notation.	Moderate	1. Describe relevant mathematical concepts and notation, and relate them to the problem-solving activity. (***PR.5.1***)	PS.1a, PS.2a, PS.2d, PS.4b	INST.PIC.2, INST.TCK.2
		2. Ask students to explain each step used to solve a problem in a worked example. (***PR.5.2***)	PS.2d, PS.3a, PS.6c, PS.7c	INST.AM.4, INST.TCK.2, PLAN.Desc.1
		3. Help students make sense of algebraic notation. (***PR.5.3***)	PS.1a, PS.2d	INST.TCK.2, PLAN.SW.3
Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students				
Recommendation 1: Use solved problems to engage students in analyzing algebraic reasoning and strategies.	Minimal	1. Have students discuss solved problem structures and solutions to make connections among strategies and reasoning. (***AK.1.1***)	PS.1a, PS.1b, PS.1c, PS.1d, PS.2d, PS.3a, PS.3b, PS.3d, PS.7b, PS.7c	INST.MS.2, INST.PIC.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.3
		2. Select solved problems that reflect the lesson's instructional aim, including problems that illustrate common errors. (***AK.1.2***)	PS.1a, PS.1b	INST.PIC.2, INST.AM.1, INST.TCK.2, PLAN.SW.3
		3. Use whole-class discussions, small-group work, and independent practice activities to introduce, elaborate on, and practice working with solved problems. (***AK.1.3***)	PS.1b, PS.1c, PS.1d, PS.2d, PS.3a, PS.3b	INST.PIC.2, INST.AM.4, INST.AM.7, INST.GS.1
Recommendation 2: Teach students to utilize the structure of algebraic representations.	Minimal	1. Promote the use of language that reflects mathematical structure. (***AK.2.1***)	PS.3a, PS.3b, PS.6c	INST.PIC.3, INST.PIC.4, INST.TCK.2, PLAN.SW.2
		2. Encourage students to use reflective questioning to notice structure as they solve problems. (***AK.2.2***)	PS.1c, PS.1d, PS.3a, PS.3d, PS.7b, PS.7c	INST.MS.1, INST.MS.2, INST.AM.4, INST.AM.5, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1
		3. Teach students that different algebraic representations can convey different information about an algebra problem. (***AK.2.3***)	PS.1c, PS.2b, PS.2d, PS.4b, PS.7b	INST.PIC.2, INST.AM.4, INST.TCK.2

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
Recommendation 3: Teach students to intentionally choose from alternative algebraic strategies when solving problems.	Moderate	1. Teach students to recognize and generate strategies for solving problems. (***AK.3.1***)	PS.1b, PS.1c, PS.3a, PS.7b	INST.MS.2, INST.PIC.2, INST.AM.4, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1
		2. Encourage students to articulate the reasoning behind their choice of strategy and the mathematical validity of their strategy when solving problems. (***AK.3.2***)	PS.1a, PS.1b, PS.1c, PS.1d, PS.3a, PS.6c, PS.7c	INST.AM.4, INST.AM.7, INST.TCK.2, INST.PS.1, PLAN.SW.1, PLAN.SW.2, PLAN.Desc.1
		3. Have students evaluate and compare different strategies for solving problems. (***AK.3.3***)	PS.1a, PS.1b, PS.1c, PS.1d, PS.3b, PS.3d, PS.7b, PS.7c	INST.MS.2, INST.AM.4, INST.AM.5, INST.AM.7, INST.AM.9, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1
Developing Effective Fractions Instruction for Kindergarten Through 8th Grade				
Recommendation 1: Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.	Minimal	1. Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects. ***(FR.1.1)***	2.ATO.3	INST.MS.2, PLAN.SW.3
		2. Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions. ***(FR.1.2)***	3.NSF.1a, 3.NSF.2a, 3.NSF.2b	PLAN.SW.3
		3. Build on students' informal understanding to develop a more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions. ***(FR.1.3)***	3.NSF	PLAN.SW.3
Recommendation 2: Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use	Moderate	1. Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share. (***FR.2.1***)	PS.1a, PS.2a, PS.2b	INST.PIC.2, INST.TCK.2, PLAN.SW.3
		2. Provide opportunities for students to locate and compare fractions on number lines. (***FR.2.2***)	PS.2a, PS.2b	INST.PIC.2

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.		3. Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions. (***FR.2.3***)	PS.1a, PS.2a, PS.2b	INST.PIC.2, INST.TCK.2, PLAN.SW.3
		4. Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms. (***FR.2.4***)	PS.1a, PS.2a, PS.2b, PS.6b 6.NS.1, 6.NS.9, 6.RP.3e, 7.NS.5, 8.NS.3	INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.3
Recommendation 3: Help students understand why procedures for computations with fractions make sense.	Moderate	1. Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures. (***FR.3.1***)	PS.1a, PS.1b, PS.1c, PS.2b, PS.2c, PS.4a, PS.4b 6.NS.1, 6.RP.3b	INST.PIC.2, INST.TCK.2, PLAN.SW.3
		2. Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions. (***FR.3.2***)	PS.1d, PS.2a, PS.4c	INST.MS.2, INST.AM.4, INST.AM.5, INST.PS.1
		3. Address common misconceptions regarding computational procedures with fractions. (***FR.3.3***)	PS.1a, PS.1b, PS.2d	INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.1, PLAN.SW.3, PLAN.Desc.1
		4. Present real-world contexts with plausible numbers for problems that involve computing with fractions. (***FR.3.4***)	PS.1a, PS.2a 6.RP.3, 7.RP.3, 8.EEI.5	INST.MS.1, INST.PIC.2, INST.AM.6, PLAN.SW.3
Recommendation 4: Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.	Minimal	1. Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems. (***FR.4.1***)	PS.1a, PS.2a 6.RP.1, 6.RP.2, 7.RP.2a	INST.PIC.2, INST.TCK.2, PLAN.SW.3
		2. Encourage students to use visual representations to solve ratio, rate, and proportion problems. (***FR.4.2***)	PS.1a, PS.1c, PS.2b, PS.4a, PS.5a 6.RP.2a, 6.RP.3b, 7.RP.2, 8.EEI.5c	INST.PIC.2, INST.TCK.2

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
		3. Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems. (***FR.4.3***)	PS.1b, PS.1c, PS.1d, PS.2c, PS.2d, PS.3b, PS.3d, PS.5a, PS.7b 7.RP.2, 7.RP.2d	INST.MS.2, INST.AM.4, INST.AM.9, INST.TCK.2, PLAN.SW.1, PLAN.Desc.1
Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools				
Recommendation 4: Interventions should include instruction on solving word problems that is based on common underlying structures.	Strong	1. Teach students about the structure of various problem types, how to categorize problems based on structure, and how to determine appropriate solutions for each problem type. (***RTI.4.1***)	PS.1b, PS.1c, PS.7c	INST.PIC.2, INST.AM.4, INST.TCK.2
		2. Teach students to recognize the common underlying structure between familiar and unfamiliar problems and to transfer known solution methods from familiar to unfamiliar problems. (***RTI.4.2***)	PS.1b, PS.1c, PS.4b, PS.7c	INST.MS.2, INST.PIC.2, INST.AM.4, INST.TCK.2, INST.TH.2, PLAN.SW.1, PLAN.SW.3
Recommendation 5: Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.	Moderate	1. Use visual representations such as number lines, arrays, and strip diagrams. (***RTI.5.1***)	PS.2a, PS.2b, PS.4a	INST.PIC.2, INST.TCK.2
		2. If visuals are not sufficient for developing accurate abstract thought and answers, use concrete manipulatives first. Although this can also be done with students in upper elementary and middle school grades, use of manipulatives with older students should be expeditious because the goal is to move toward understanding of—and facility with—visual representations, and finally, to the abstract. (***RTI.5.2***)	PS.1a, PS.2a, PS.2b, PS.4a	INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.3
Recommendation 6: Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.	Moderate	1. Provide about 10 minutes per session of instruction to build quick retrieval of basic arithmetic facts. Consider using technology, flashcards, and other materials for extensive practice to facilitate automatic retrieval. (***RTI.6.1***)		INST.AM.10, INST.TCK.2
		2. Teach students in grades 2 through 8 how to use their knowledge of properties, such as commutative, associative, and distributive law, to derive facts in their heads. (***RTI.6.2***)	PS.1a, PS.2a	INST.AM.4, INST.TCK.2, PLAN.SW.3

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
Teaching Math to Young Children				
Recommendation 1: Teach numbers and operations using a developmental progression.	Moderate	1. First, provide opportunities for children to practice recognizing the total number of objects in small collections (one to three items) and labeling them with a number word without needing to count them. ***(YC.1.1)***	K.NS.6	No direct alignment
		2. Next, promote accurate one-to-one counting as a means of identifying the total number of items in a collection. ***(YC.1.2)***	K.NS.5	PLAN.SW.3
		3. Once children can recognize or count collections, provide opportunities for children to use number words and counting to compare quantities. ***(YC.1.3)***	K.NS.7	PLAN.IP.3, PLAN.SW.3
		4. Encourage children to label collections with number words and numerals. ***(YC.1.4)***	K.NS.4a	PLAN.SW.3
		5. Once children develop these fundamental number skills, encourage them to solve basic problems. ***(YC.1.5)***	K.PS.1, K.PS.4, K.ATO	PLAN.IP.3, PLAN.SW.3
Recommendation 2: Teach geometry, patterns, measurement, and data analysis using a developmental progression.	Minimal	1. Help children recognize, name, and compare shapes, and then teach them to combine and separate shapes. ***(YC.2.1)***	K.G.2, K.G.4, 1.G.2, 1.G.3, 1.G.4	No direct alignment
		2. Encourage children to look for and identify patterns, and then teach them to extend, correct, and create patterns. ***(YC.2.2)***	PS.7, 1.ATO.9a, 1.ATO.9b	No direct alignment
		3. Promote children's understanding of measurement by teaching them to make direct comparisons and to use both informal or nonstandard (e.g., the child's hand or foot) and formal or standard (e.g., a ruler) units and tools. ***(YC.2.3)***	K.MDA.2, 1.MDA.2, 2.MDA.1	INST.MS.2, PLAN.SW.3
		4. Help children collect and organize information, and then teach them to represent that information graphically. ***(YC.2.4)***	PS.4, K.MDA.3, K.MDA.4, 1.MDA.4	INST.MS.2

WWC Practice Guide *Recommendation*	Level of Evidence	Instructional Strategy (***Strategy code***)	SC Standards Alignment	
			Mathematics	Teacher
Recommendation 3: Use progress monitoring to ensure that math instruction builds on what each child knows.	Minimal	1. Use introductory activities, observations, and assessments to determine each child's existing math knowledge, or the level of understanding or skill they have reached on a developmental progression. ***(YC.3.1)***	No direct alignment	INST.MS.2, PLAN.IP.3, PLAN.SW.3
		2. Tailor instruction to each child's needs, and relate new ideas to their existing knowledge. ***(YC.3.2)***	PS.1a, PS.2a	PLAN.IP.3, PLAN.SW.3
		3. Assess, record, and monitor each child's progress so that instructional goals and methods can be adjusted as needed. ***(YC.3.3)***	No direct alignment	PLAN.IP.3, PLAN.SW.3, PLAN.A.2
Recommendation 4: Teach children to view and describe their world mathematically.	Minimal	1. Encourage children to use informal methods to represent math concepts, processes, and solutions. ***(YC.4.1)***	K.G.4, 1.MDA.2	INST.MS.2
		2. Help children link formal math vocabulary, symbols, and procedures to their informal knowledge or experiences. ***(YC.4.2)***	PS.1a, PS.4	PLAN.SW.3
		3. Use open-ended questions to prompt children to apply their math knowledge. ***(YC.4.3)***	PS.1, PS.2, PS.3	INST.MS.2
		4. Encourage children to recognize and talk about math in everyday situations. ***(YC.4.4)***	PS.2a	INST.MS.2
Recommendation 5: Dedicate time each day to teaching math, and integrate math instruction throughout the school day.	Minimal	1. Plan daily instruction targeting specific math concepts and skills. ***(YC.5.1)***	No direct alignment	PLAN.IP.3, PLAN.A.2
		2. Embed math in classroom routines and activities. ***(YC.5.2)***	No direct alignment	INST.MS.2
		3. Highlight math within topics across the curriculum. ***(YC.5.3)***	No direct alignment	INST.MS.2, PLAN.SW.3
		4. Create a math-rich environment where children can recognize and meaningfully apply math. ***(YC.5.4)***	No direct alignment	INST.MS.2
		5. Use games to teach math concepts and skills, and to give children practice in applying them. ***(YC.5.5)***	No direct alignment	No direct alignment

This document provides a summary of Recommendation 1 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Minimal**

RECOMMENDATION

Prepare problems and use them in whole-class instruction.

Teachers should set aside time for problem-solving activities with the entire class instead of limiting problem-solving to individual homework assignments and include a variety of problems in these activities. Additionally, teachers should ensure that students understand the language, context, and math concepts of the problems included in lessons and homework.

HOW TO CARRY OUT THE RECOMMENDATION

1. Include both routine and nonroutine problems in problem-solving activities.

Instructional strategies from the examples

- Align use of routine or nonroutine problems with students' previous experience with problem-solving.
- Routine problems can be solved using familiar methods, replicating previously learned methods in a step-by-step fashion.

South Carolina standards alignment

MATHEMATICS: PS.1c
TEACHERS: INST.MS.1, INST.AM.4, INST.TCK.2, PLAN.SW.1

Nonroutine problems involve approaches that are not as predictable or well rehearsed; solution pathways aren't explicitly suggested by the task, task instructions, or in a worked-out example. Routine problems can be solved using approaches that students have already learned. Nonroutine problems, on the other hand, require using approaches that students are less familiar with or that are less obvious from the problem. When the goal of a lesson is to help students understand the meaning of an operation or mathematical idea, teachers should select routine problems. These do not necessarily have to be simple—they can be complex, multistep problems that involve problem-solving approaches students are already familiar with. When the goal of a lesson is to develop students' ability to think strategically, teachers should select nonroutine problems.

Examples of routine problems

Likely routine for a student who has studied and practiced multiplication with mixed numbers:

Carlos is following a cookie recipe that calls for $1\,{}^{2}/_{3}$ cups of flour.
He needs to make 4 batches of cookies. How much flour does he need?
Likely routine for a student who has studied and practiced solving linear equations with one variable:

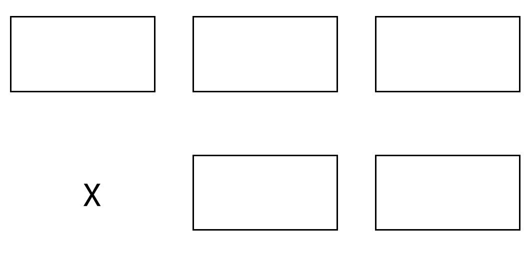
Solve for x: $20 + 8x = 60$

Note. Adapted from Example 1 on page 12 of the practice guide.

Examples of nonroutine problems

Likely nonroutine for students who are solidifying their understanding of multiplication:

The digits 1, 2, 3, 4, and 5—using each of these digits only once—are arranged in the blank boxes in the template below. Of all the possible arrangements of the digits 1 through 5, which one will produce the largest possible product? Why does this arrangement work?



Likely nonroutine for students in beginning algebra:

There are 20 people in a room. Everybody shakes the hand of everyone else. How many handshakes occurred?

Note. Adapted from Example 2 on page 12 of the practice guide.

2. Ensure that students will understand the problem by addressing issues students might encounter with the problem's context or language.

Instructional strategies from the examples

- Explain contexts or vocabulary that may be unfamiliar to ensure students understand the language and context of problems—not to make problems less challenging, but to allow students to focus on the mathematics in the problem rather than on the need to learn new background knowledge or language.

South Carolina standards alignment

MATHEMATICS: PS.2d
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.Desc.1

The problems a teacher selects for a lesson may include unfamiliar vocabulary or contexts, making it challenging for students to focus on the math content. This is a particularly critical issue for English learners and students with disabilities. To ensure students' understanding without lessening the mathematical challenge, teachers can:

- Choose problems with language or contexts that are appropriate for the students' background.
- Clarify unfamiliar language or contexts in existing problems.
- Reword problems that contain unfamiliar words or phrases for students.

Examples of clarifying vocabulary and context

Example Problem	Vocabulary	Context
In a factory, 54,650 parts were made. When they were tested, 4% were found to be defective. How many parts were working?	Students need to understand the term defective as being the opposite of working and the symbol % as percent to correctly solve the problem.	What is a factory? What does parts mean in this context?

At a used-car dealership, a car was priced at $7,000. The salesperson then offered a discount of $350. What percent discount, applied to the original price, gives the offered price?	Students need to know what offered and original price mean to understand the goal of the problem, and they need to know what discount and percent discount mean to understand what mathematical operators to use.	What is a used-car dealership?

Note. Taken from Example 3 on page 14 of the practice guide.

3. Consider students' knowledge of math content when planning lessons.

Instructional strategies from the examples

- Review relevant skills and knowledge needed to understand and solve a problem, especially if the mathematical content has not been discussed recently or if a nonroutine, challenging problem is presented

South Carolina standards alignment

MATHEMATICS: PS.1c
TEACHERS: INST.MS.1, INST.AM.4, INST.TCK.2, PLAN.SW.1

Teachers should consider the concepts, skills, and vocabulary their students will need to solve problems included in lessons. For example, when finding the area of a circle, students may need to review the definitions of radius and pi as well as the concepts of perimeter and area. A brief review of the skills and vocabulary needed to understand and solve a problem may not only benefit struggling students but also help all students see how the knowledge they already have applies to more challenging problems.

Example of reviewing mathematical language

Problem	
Two vertices of a triangle are located at (0, 4) and (0, 10). The area of the triangle is 12 square units. What are all possible positions for the third vertex?	
Mathematical Language to Review	
• Vertices • Area square units	• Triangle • Vertex

Note. Taken from Example 5 on page 15 of the practice guide.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
Teachers are having trouble finding problems for the problem-solving activities.	Teachers can reference supplementary materials (for example, books on problem-solving), ask colleagues for additional problem-solving activities, or search the internet for examples. Useful resources on the internet include "Problems of the Week" from the Math Forum (https://www.nctm.org/pows/), "Illuminations" from the National Council of Teachers of Mathematics (https://illuminations.nctm.org/), and practice problems from standardized tests such as the PISA (http://www.oecd.org/pisa/; PISA 2012 Mathematics Items), SAT(https://collegereadiness.collegeboard.org/sat/practice), or TIMSS (TIMSS—Released Assessment Questions).
Teachers have no time to add problem-solving activities to their math instruction.	To make time during lessons, teachers can replace some of the problems students are required to solve during seatwork with fully solved problems that students can review and use as problem-solving models.
Teachers are not sure which words to teach when teaching problem-solving.	Math coaches and specialists can provide lists of words and phrases essential for teaching a given unit. Teachers can also work with colleagues to identify words students need to understand and solve problems. They can also look for important terms in class textbooks or state math standards.

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16

This document provides a summary of Recommendation 2 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Strong**

RECOMMENDATION

Assist students in monitoring and reflecting on the problem-solving process.

Considering what students are doing and why they are doing it during the problem-solving process helps them learn math more effectively. It helps students assess the path they take while solving a problem and helps them connect knowledge they already have to newer concepts. Teachers can help students in this process through asking guiding questions, modeling self-monitoring and reflection, and building on students' own reflections to help them improve their problem-solving.

HOW TO CARRY OUT THE RECOMMENDATION

1. Provide students with a list of prompts to help them monitor and reflect during the problem-solving process.

Instructional strategies from the examples

- Provide questions that students should ask and answer as they solve problems.
- Provide task lists that help students complete steps in the problem-solving process.
- Encourage students to explain and justify their response to each prompt, either orally or in writing.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c
TEACHERS: INST.AM.5, INST.TCK.2, INST.PS.1, PLAN.SW.1

Teachers can provide students with two types of prompts: (1) questions students should ask themselves, and (2) task lists that students can follow each time they work through solving a problem. When first introducing prompts, teachers may need to help students understand how to use them. Teachers can help students by talking through the questions or task lists in whole-class or small-group discussions. If students obtain an incorrect solution, teachers can provide the correct solution and ask students to explain why that answer is correct (and, conversely, why their original solution is wrong). The more competent students become at problem-solving, the less support teachers will need to give. Teachers should be sure not to overburden students with long lists of prompts. Doing so may lead students to solve problems more slowly or abandon the prompts altogether.

Example Questions	**Example Task List**
• What is this problem asking? • What do I know about the problem so far? What information is given to me? How can this help me? • Which information in this prompt is relevant to solving the problem? • What are some different ways I could approach solving this problem? • Why did this approach work? Why didn't it work?	1. Figure out what the question is asking for and what information is given. 2. Identify the type of problem. 3. Recall solutions to previous problems that may be useful in the current problem. 4. Make a visual to help represent or solve the problem. 5. Solve the problem. 6. Check your solution.

Note. Adapted from Examples 6 and 7 on page 19 of the practice guide.

2. Model how to monitor and reflect on the problem-solving process.

Instructional strategies from the examples

- As you work through a problem, say not only the response to each prompt but also the reason(s) why you took each.
- Alternatively, as you work a step in solving the problem, ask students to explain why this works.

South Carolina standards alignment

MATHEMATICS: PS.1c, PS.3d
TEACHERS: INST.PIC.3, INST.AM.5, INST.PS.1, PLAN.SW.1

Using prompts, teachers can show how to monitor and reflect during the problem-solving process. Teachers can give students an appropriate response to each prompt and either explain the reasoning behind that response or ask the students to explain why that response makes sense. Teachers should ensure that each step of the problem-solving process is represented by a prompt.

Example of modeling how to monitor and reflect

Problem
Last year was unusually dry in Colorado. Denver usually gets 60 inches of snow per year. Vail, which is up in the mountains, usually gets 350 inches of snow. Both places had 10 inches of snow less than the year before. Kara and Ramon live in Colorado and heard the weather report. Kara thinks the decline for Denver and Vail is the same. Ramon thinks that when you compare the two cities, the decline is different. Explain how both people are correct.

Solution
TEACHER: First, I ask myself, "What is this story about, and what do I need to find out?" I see that the problem has given me the usual amount of snowfall and the change in snowfall for each place, and that it talks about a decline in both cities. I know what decline means: "a change that makes something less." Now I wonder how the decline in snowfall for Denver and Vail can be the same for Kara and different for Ramon. I know that a decline of 10 inches in both cities is the same, so I guess that's what makes Kara correct. How is Ramon thinking about the problem? I ask myself, "Have I ever seen a problem like this before?" As I think back to the assignments we had last week, I remember seeing a problem that asked us to calculate the discount on a $20 item that was on sale for $15. I remember we had to determine the percent change. This could be a similar kind of problem. This might be the way Ramon is thinking about the problem. Before I go on, I ask myself, "What steps should I take to solve this problem?" It looks like I need to divide the change amount by the original amount to find the percent change in snowfall for both Denver and Vail. **Denver:** $10 \div 60 = 0.166$ or 16.67% or 17% when we round it to the nearest whole number **Vail:** $10 \div 350 = 0.029$ or 2.9% or 3% when we round it to the nearest whole number So the percent decrease in snow for Denver was much greater (17%) than for Vail (3%). Now I see what Ramon is saying! It's different because the percent decrease for Vail is much smaller than it is for Denver. Finally, I ask myself, "Does this answer make sense when I reread the problem?" Kara's answer makes sense because both cities did have a decline of 10 inches of snow. Ramon is also right because the percent decrease for Vail is much smaller than it is for Denver. Now, both of their answers make sense to me.

Note. Taken from Example 8 on page 20 of the practice guide.

3. Use student thinking about a problem to develop students' ability to monitor and reflect.

Instructional strategies from the examples

- Help students verbalize other ways to think about the problem.
- Include guided questioning to help students clarify and refine their thinking or establish a method for monitoring and reflecting that makes sense to them.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1c, PS.3a
TEACHERS: INST.MS.1, INST.MS.2, INST.AM.5, INST.PS.1, PLAN.SW.1, PLAN.Desc.1

Teachers can help students establish methods for monitoring and reflecting that make sense to students. Teachers can establish a dialogue with students that includes guiding questions to clarify and refine their thinking. This activity is helpful for students who dislike, or have trouble understanding, teacher-provided prompts.

Example of using student ideas to clarify and refine the monitoring and reflecting process

Problem
Find a set of five different numbers whose average is 15.
Solution
TEACHER: Jennie, what did you try? **STUDENT:** I'm guessing and checking. I tried 6, 12, 16, 20, 25, and they didn't work. The average is like 17.8 or something decimal like that. **TEACHER:** That's pretty close to 15, though. Why'd you try those numbers? **STUDENT:** What do you mean? **TEACHER:** I mean, where was the target, 15, in your planning? It seems like it was in your thinking somewhere. If I were choosing five numbers, I might go with 16, 17, 20, 25, 28. **STUDENT:** But they wouldn't work—you can tell right away.

TEACHER: How?

STUDENT: Because they are all bigger than 15.

TEACHER: So?

STUDENT: Well, then the average is going to be bigger than 15.

TEACHER: Okay. That's what I meant when I asked, "Where was 15 in your planning?" You knew they couldn't all be bigger than 15. Or they couldn't all be smaller either?

STUDENT: Right.

TEACHER: Okay, so keep the target, 15, in your planning. How do you think five numbers whose average is 15 relate to the number 15?

STUDENT: Well, some have to be bigger and some smaller. I guess that's why I tried the five numbers I did.

TEACHER: That's what I guess, too. So the next step is to think about how much bigger some have to be and how much smaller the others have to be. Okay?

STUDENT: Yeah.

TEACHER: So use that thinking to come up with five numbers that work.

Note. Taken from Example 9 on page 21 of the practice guide.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Students don't want to monitor and reflect; they just want to solve the problem.	Explain to students that getting into the habit of monitoring and reflecting every time they solve a problem will improve their problem-solving abilities, and that monitoring and reflecting are still an integral part of the process for experienced problem-solvers.
Teachers are unclear on how to think aloud while solving a nonroutine problem.	Teachers can prepare ahead of time by creating outlines of responses to prompts and by anticipating how students may think about prompts. They can seek help from colleagues or math coaches if they get stuck.
Students take too much time to monitor and reflect on the problem-solving process.	While students may solve problems slowly when they begin learning how to monitor and reflect, they will become more efficient with practice.
When students reflect on the problems they have already solved, they resort to using methods from problems rather than adapting their efforts to the new problem before them.	Ask students to explain why their solution worked in the previous problem, and why it may or may not work for the current problem.

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16

This document provides a summary of Recommendation 3 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Strong**

Recommendation

Teach students how to use visual representations.

Translating quantitative information into an algebraic or arithmetic form is a critical component of the problem-solving process. Students who learn how to represent this information visually before translating it into an algebraic or arithmetic form tend to be more effective at problem-solving. When teaching students to use visual representations (e.g., graphs, diagrams, number lines, tables), teachers should choose visuals that are appropriate for the problem at hand and their students, then use them consistently for similar problems so as not to overwhelm students with too many differing examples.

Sample visual representations

Strip diagrams use rectangles to represent quantities presented in the problem.

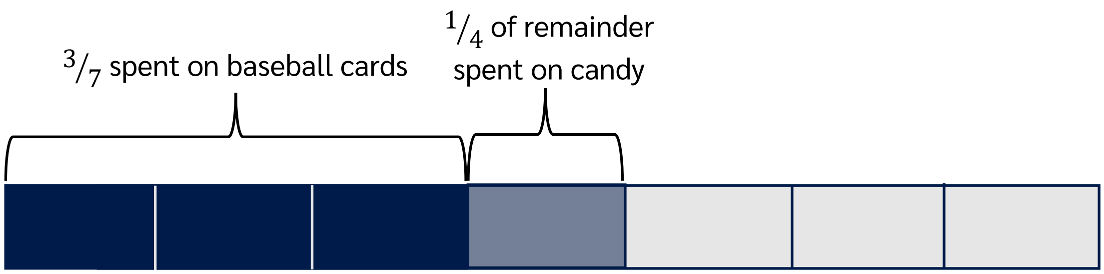
Percent bars are strip diagrams in which each rectangle represents a part of 100 in the problem.

Schematic diagrams demonstrate the relative sizes and relationships between quantities in the problem.

Below are two examples of how visual representations might be used to solve problems. For additional examples, see page 24 of the practice guide.

Problem
Ben spent $^{3}/_{7}$ of his allowance on baseball cards and then $^{1}/_{4}$ of what remained on candy. After this, he had \$50 left. How much did he start with?

Sample Strip Diagram



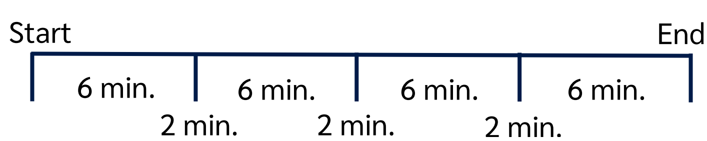
This diagram depicts the original amount of money that Ben had, divided into 7 equal parts. It can be seen that 3 of the 7 parts have not been spent. From this diagram, there are multiple approaches to creating an equation.

Note. Adapted from Example 10 on page 24 of the practice guide.

Problem

Jackie usually jogs 4 laps around a track, and each lap takes 6 minutes.
Because of injury, she needs to rest for 2 minutes between each lap.
How long does it take her to complete the 4 laps?

Sample Schematic Diagram



This diagram illustrates Jackie running 4 laps, with each lap taking 6 minutes. As she runs, Jackie needs to take a 2-minute break between each lap. From this diagram, an equation, such as $(6 \times 4) + (2 \times 3) = x$, can be created to find the total number of minutes Jackie takes to run 4 laps.

Note. Adapted from Example 10 on page 25 of the practice guide.

How to carry out the recommendation

1. Select visual representations that are appropriate for students and the problems they are solving.

Instructional strategies from the examples

- Use schematic diagrams with ratio and proportion problems.
- Use percent bars for percent problems
- Use strip diagrams for comparison and fraction problems.

South Carolina standards alignment

MATHEMATICS: PS.2b
TEACHERS: INST.PIC.2, INST.AM.6, INST.TCK.2

Rather than using all visual representations recommended for a particular type of problem, teachers should select the visual representations they think will work best for their students. Teachers should use selected representations consistently for similar problems so as not to overwhelm students with too many examples and give students time to learn how to successfully use the selected representations. If students still struggle with a representation after a reasonable amount of time, teachers should consider using a different type of representation.

2. Use think-alouds and discussions to teach students how to represent problems visually.

Instructional strategies from the examples

- Demonstrate how to represent the problem using the representation, using think-alouds to describe the decisions you make to connect the problem to the representation.
- Explain how to identify the problem type based on mathematical ideas in the problem and why a certain representation is most appropriate.
- Teach students to identify what information is relevant or critical to solving the problem.
- Encourage students to discuss similarities or differences among visuals they have used.

South Carolina standards alignment

MATHEMATICS: PS.1c, PS.2b, PS.4a, PS.7c
TEACHERS: INST.MS.2, INST.PIC.2, INST.PIC.3, INST.TCK.2, INST.PS.

Teachers should demonstrate the thought process behind connecting a problem to a visual representation by thinking aloud when explaining a new representation. Thinking aloud goes beyond teachers telling students what they are doing; it involves teachers explaining why they are taking the particular steps. Teachers should explain how they identified the type of math problem and why they think the selected representation is appropriate for that problem. They should demonstrate how to identify the information in a problem that is relevant to solving it.

Example of using a think-aloud

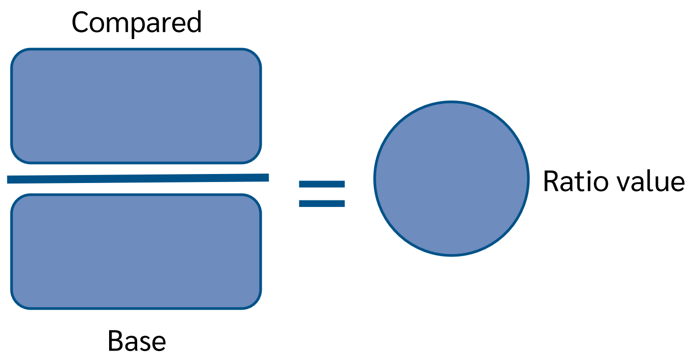
Problem
Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

Solution

TEACHER: I know this is a ratio problem because two quantities are being compared: the number of red roses and the number of yellow roses. I also know the ratio of the two quantities. There are 3 red roses for every 5 yellow roses. This tells me I can find more of each kind of rose by multiplying.

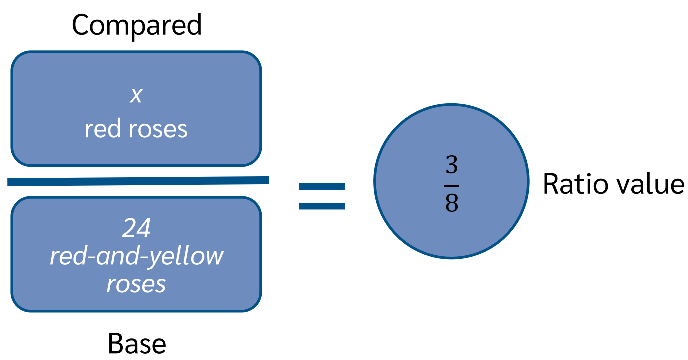
I reread the problem and determine that I need to solve the question posed in the last sentence: "How many red roses are in Monica's bouquet?" Because the question is about Monica, perhaps I don't need the information about Bianca. The third sentence says there are two dozen red and yellow roses. I know that makes 24 red and yellow roses, but I still don't know how many red roses there are. I know there are 3 red roses for every 5 yellow roses. I think I need to figure out how many red roses there are in the 24 red and yellow roses.

Let me reread the problem . . . That's correct. I need to find out how many red roses are in the bouquet of 24 red and yellow roses. The next part of the problem talks about the ratio of red roses to red and yellow roses. I can draw a diagram that helps me understand the problem. I've done this before with ratio problems. These kinds of diagrams show the relationship between the two quantities in the ratio.



TEACHER: I write the quantities and units from the problem and an x for what must be solved in the diagram. First, I am going to write the ratio of red roses to yellow roses here in the circle. This is a part-to-whole comparison—but how can I find the whole in the part-to-whole ratio when we only know the part-to-part ratio (the number of red roses to the number of yellow roses)?

I have to figure out what the ratio is of red roses to red and yellow roses when the problem only tells me about the ratio of red roses to yellow roses, which is 3:5. So if there are 3 red roses for every 5 yellow roses, then the total number of units for red and yellow roses is 8. For every 3 units of red roses, there are 8 units of red and yellow roses, which gives me the ratio 3:8. I will write that in the diagram as the ratio value of red roses to red and yellow roses. There are two dozen red and yellow roses, and that equals 24 red and yellow roses, which is the base quantity. I need to find out how many red roses (x) there are in 24 red and yellow roses.



I can now translate the information in this diagram to an equation like this:

$$\frac{x\ red\ roses}{24\ red-and-yellow\ roses} = \frac{3}{8}$$

Then, I need to solve for x.

$$\frac{x}{24} = \frac{3}{8}$$

$$24\left(\frac{x}{24}\right) = 24\left(\frac{3}{8}\right)$$

$$x = \frac{72}{8}$$

$$x = 9$$

Note. Taken from Example 11 on pages 27–28 of the practice guide.

3. Show students how to convert the visually represented information into mathematical notation.

Instructional strategies from the examples

- Show students how each quantity and relationship in the visual representation corresponds to those in the equation.

South Carolina standards alignment

MATHEMATICS: PS.1c, PS.2a, PS.2b, PS.4a, PS.4b
TEACHERS: INST.PIC.2, INST.TCK.2

Teachers should show students how to translate quantities and relationships in visual representations into math equations. Sometimes, all this translation requires is removing boxes, arrows, and other visual elements from the representation. With more complicated examples, teachers may need to provide more explicit illustrations of the connection between the representations and mathematical notation.

Potential roadblocks and how to address them

Roadblock	Suggested **Approach**
Students do not capture the relevant details in the problem or include unnecessary details when representing a problem visually.	If students are missing relevant details in their visual representations, teachers can ask guiding questions to build on students' thinking and refine their representations. Once the representations are refined, teachers can ask students why their initial representations did not work. If guiding questions do not work, teachers can demonstrate how to alter students' representations to represent the problems appropriately and eliminate unnecessary detail. Teachers should point out elements of the representations that were done correctly so students are encouraged to continue trying.
The class text does not use visual representations.	Teachers can incorporate visual representations into lessons using media such as whiteboards, overhead projectors, or interactive whiteboards. Teachers can tap colleagues or math coaches for useful visual representations or develop their own. The internet and professional development materials may have useful examples.

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16

This document provides a summary of Recommendation 4 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Expose students to multiple problem-solving strategies.

Students who learn how to use multiple problem-solving strategies can approach problems with greater ease and flexibility in finding solutions. Teachers should demonstrate that problems can be solved in multiple ways and that approaches to problem-solving should be selected for their ease and efficiency. Teachers can present students with a number of problem-solving strategies and then give them the opportunity to compare, contrast, and carry out the strategies.

How to carry out the recommendation

1. Provide instruction in multiple strategies.

Instructional strategies from the examples

- Demonstrate multiple ways to solve the same problem.
- At times, use an unsuccessful strategy and demonstrate changing to an alternate strategy.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c
TEACHERS: INST.PIC.2, INST.AM.9, INST.TCK.2, INST.PS.1

Teachers can present strategies for general use as well as those for specific problems. When demonstrating different strategies, teachers should occasionally attempt unsuccessful strategies and then change to successful ones. Doing so will show students that some problems may not be easy to solve the first time and they may need to try more than one strategy before successfully solving a problem. Teachers should also demonstrate more than one successful approach to the same problem.

Example of two ways to solve the same problem

Problem

Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Solution 1: Guess and Check

4 x 4 legs = 16 legs	9 x 3 legs = 27 legs	Total = 43 legs
3 x 4 legs = 12 legs	8 x 3 legs = 24 legs	Total = 36 legs
2 x 4 legs = 8 legs	7 x 3 legs = 21 legs	Total = 29 legs

TEACHER: This works; the total equals 29, and with two 4-legged stools and seven 3-legged stools, there are five more 3-legged stools than 4-legged stools.

Solution 2

TEACHER: Let's see if we can solve this problem logically. The problem says that there are five more 3-legged stools than 4-legged stools. It also says that there are 29 legs altogether. If there are five more 3-legged stools, there has to be at least one 4-legged stool in the first place. Let's see what that looks like.

Stools			
Total legs	$4 \times 1 = 4$	+	$3 \times 6 = 18$
		$4 + 18 = 22$	

TEACHER: We can add a stool to each group, and there will still be a difference of five stools.

Stools			
Total legs	$4 \times 2 = 8$	+	$3 \times 7 = 21$
		$8 + 21 = 29$	

TEACHER: I think this works. We have a total of 29 legs, and there are still five more 3-legged stools than 4-legged stools. We solved this by thinking about it logically. We knew there was at least one 4-legged stool and there were six 3-legged stools. Then we added to both sides so we always had a difference of five stools.

Note. Taken from Example 14 on page 34 of the practice guide.

2. Provide opportunities for students to compare multiple strategies in worked examples.

Instructional strategies from the examples

- Ask students to compare the similarities and differences among multiple strategies.
- Provide opportunities for students to work with a partner to discuss strategies in worked examples.
- Use worked examples alongside opportunities for students to solve problems on their own.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.3b, PS.3d, PS.7b, PS.7c
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.1

Teachers should provide side-by-side examples of different problem-solving strategies and give students the opportunity to work together to compare the strategies. Teachers can prompt students to compare and contrast the strategies, to justify the approach they would choose, and to consider why two different approaches can lead to the same answer. Teachers can provide worked examples alongside problems that students are required to solve. Students can practice describing their solution paths both verbally and in writing.

Example of comparing strategies

Sanjin's Solution		Emily's Solution	
$7(x-3)=4(x-3)-3$		$7(x-3)=4(x-3)-3$	
$7x-21=4x-12-3$	Distribute	$3(x-3)=-3$	Subtract on both
$7x-21=4x-15$	Combine	$x-3=-1$	Divide on both
$3x-21=-15$	Subtract on both	$x=2$	Add on both
$3x=6$	Add on both		
$x=2$	Divide on both		
TEACHER: Sanjin and Emily used different approaches but got the same answer. Why is this? Which of their approaches would you choose? Why?			

Note. Adapted from Example 15 on page 35 of the practice guide.

3. Ask students to generate and share multiple strategies for solving a problem.

Instructional strategies from the examples

- Rather than randomly calling on students to share their strategies, select students purposefully based on the strategies they have used to solve the problem.

South Carolina standards alignment

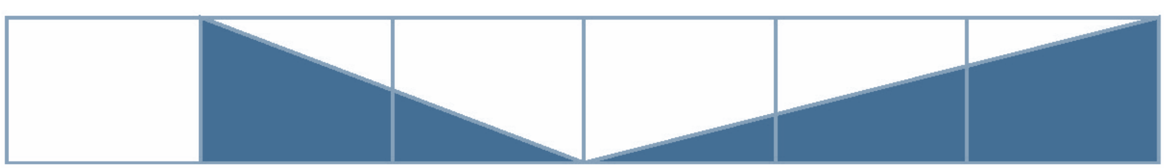
MATHEMATICS: PS.1b, PS.1c, PS.3a, PS.5a, PS.7b, PS.7c
TEACHERS: INST.MS.1, INST.MS.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1

Teachers should encourage students to generate multiple problem-solving strategies and give them opportunities to share their strategies with the class. Rather than calling on students randomly, teachers should call on students who generated strategies different from the one presented. Teachers should encourage students to not only present their approach but also to explain why they chose that approach. Examples 16 and 17 on pages 36–37 of the practice guide provide excellent models for teachers.

Example of two students sharing their strategies for solving a fractions problem

Problem

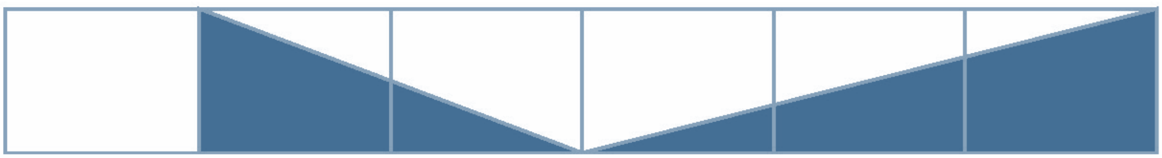
What fraction of the whole rectangle is blue?



Solution

STUDENT 1: If I think of it as what's to the left of the middle plus what's to the right of the middle, then I see that on the left, the blue part is $1/3$ of the area, so that is $1/3$ of $1/2$ of the entire rectangle. On the right, the blue part is $1/2$ of the area, so it is $1/2$ of $1/2$ of the entire rectangle. This information tells me that the blue part is

$(1/3 \times 1/2) + (1/2 \times 1/2) = 1/6 + 1/4 = 2/12 + 3/12 = 5/12$ of the entire rectangle.



STUDENT 2: I see that the original blue part and the part I've colored dark blue have the same area. So the original blue part is $1/2$ of the dark-blue-and-blue part, or $1/2$ of $5/6$ of the entire rectangle.

This tells me that the original blue part is $1/2 \times 5/6 = 5/12$ of the entire rectangle.

Note. Taken from example 17 on page 37 of the practice guide.

Potential roadblocks and how to address them

Roadblock	*Suggested Approach*
Teachers don't have enough time in their math class for students to present and discuss multiple strategies.	Teachers can ask students to write out their strategies on personal whiteboards or chart paper so they do not have to spend time rewriting them on the class whiteboard. Another approach is for teachers to document the strategies students come up with during independent or small-group work and summarize them for the class.
Not all students are willing to share their strategies.	Teachers should encourage students to share their strategies even if they are incorrect. They should explain to students that there may be a variety of approaches to solving most problems and that the solutions they share may not have been considered by the other students. Teachers can point out that sharing will help students learn effective problem-solving strategies from one another.
Some students struggle to learn multiple strategies.	If students lack or cannot retrieve necessary knowledge, they may struggle with using multiple strategies. Teachers may need to ask students to write down the facts of a problem before trying to solve it. They may need to modify a problem to make it easier to focus on problem-solving (rather than the arithmetic, for example). Teachers can also write down problem solutions side by side so that students can more easily compare the solutions.
Some of the strategies students share are not clear or do not make sense to the class.	Teachers can walk around the classroom and ask students to explain their approaches individually. Doing so will better prepare teachers to clarify students' thinking when they share their approaches with the class, either by asking guiding questions or by rewording the students' approaches. Teachers can also ask other students to restate what a student said.

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16

This document provides a summary of Recommendation 5 from the WWC practice guide *Improving Mathematical Problem Solving in Grades 4 Through 8*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **4–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Help students recognize and articulate math concepts and notation.

Students who have a solid understanding of math concepts and notation are better able to identify the math content of a problem, adapt their current knowledge to new problems, and consider multiple approaches to solving problems. Teachers can convey math concepts and notation by using problem-solving activities, by asking students to use mathematically valid explanations to explain how a worked problem was solved, and by introducing students to algebraic notation in a systematic manner.

How to carry out the recommendation

1. Describe relevant math concepts and notation, and relate them to the problem-solving activity.

Instructional strategies from the examples

- Watch and listen for opportunities to call attention to mathematical concepts and notation students use as they solve problems.
- Draw attention to mathematical ideas and concepts by directly instructing students in them before engaging the students in problem-solving.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, PS.2d, PS.4b
TEACHERS: INST.PIC.2, INST.TCK.2

Teachers can help students learn to connect their existing math intuition to formal concepts and notation. Teachers can do this is by watching and listening while students solve problems and pointing out formal concepts and notations as they are used. If students use informal language or notation for a concept, teachers can translate it into formal mathematical language and explain that there are multiple ways to communicate the same idea, moving students toward more formal mathematical approaches. Occasionally, teachers may need to explicitly instruct students in formal math concepts and notation before proceeding to problem-solving.

Example of students' intuitive understanding of formal math concepts

Problem
Is the sum of two consecutive numbers always odd?
Solution
STUDENT: Yes. **TEACHER:** How do you know? **STUDENT:** Well, suppose you take a number, like 5. The next number is 6. For 5, I can write five lines, like this: ||||| For 6, I can write five lines and one more line next to it, like this: ||||| | Then, I can count all of them, and I get 11 lines. See? It's an odd number. **TEACHER:** When you say, "It's an odd number," you mean the sum of the two consecutive numbers is odd. So can you do that with any whole number, like n? What would the next number be? **STUDENT:** It would be $n + 1$. **TEACHER:** So can you line them up like you did for 5 and 6? **STUDENT:** You mean, like this? n $\underline{n + 1}$ **TEACHER:** Right. So what does that tell you about the sum of n and $n + 1$? **STUDENT:** It's 2 ns and 1, so it's odd. **TEACHER:** Very good. The sum, which is $n + n + 1 = 2n + 1$, is always going to be odd.

Note. Taken from example 18 on page 41 of the practice guide.

2. Ask students to explain each step used to solve a problem in a worked example.

Instructional strategies from the examples

- Provide students with opportunities to explain the process used to solve a problem in a worked example and to explain why the steps worked.
- Use small-group activities to encourage students to discuss the process used in a worked example and the reasoning for each step.
- Use probing questions to help students articulate mathematically valid explanations.

South Carolina standards alignment

MATHEMATICS: PS.2d, PS.3a, PS.6c, PS.7c
TEACHERS: INST.AM.4, INST.TCK.2, PLAN.Desc.1

Teachers should give students an opportunity to explain what problem-solving process they used in a worked example as well as why that process worked. Students can discuss the problem-solving process in small groups or restate one another's explanations in pairs. If students struggle to come up with mathematically valid explanations, teachers should ask questions to guide students to more valid explanations. Teachers also can provide examples of valid explanations or reword students' explanations.

An abridged example of student explanations

Problem
Are ${}^{2}/_{3}$ and ${}^{8}/_{12}$ equivalent fractions?
An explanation that is not mathematically valid
Student: To find an equivalent fraction, whatever we do to the top of ${}^{2}/_{3}$ we must also do to the bottom. **Teacher:** What do you mean? **Student:** It just works when you multiply it.

Teacher: What happens when you multiply in this step?

Student: The fraction . . . stays the same.

Teacher: That's right. When you multiply a numerator and denominator by the same number, you get an equivalent fraction. Why is that?

Student: Before there were 3 parts, but we made 4 times as many parts, so now there are 12 parts.

Teacher: Right, you had 2 parts of a whole of 3. Multiplying both by 4 gives you 8 parts of a whole of 12. That is the same part-whole relationship—the same fraction, as you said. Here's another way to look at it: When you multiply the fraction by $^4/_4$, you are multiplying it by a fraction equivalent to 1; this is the identity property of multiplication, and it means when you multiply anything by 1, the number stays the same.

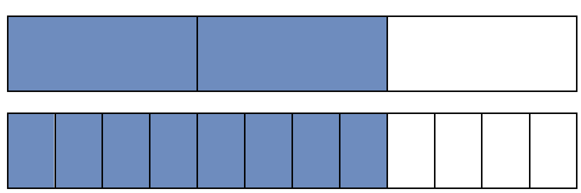
A correct description, but still not a complete explanation

Student: Whatever we multiply the top of $^2/_3$ by we must also multiply the bottom by.

A mathematically valid explanation

Student: You can get an equivalent fraction by multiplying the numerator and denominator of $^2/_3$ by the same number. If we multiply the numerator and denominator by 4, we get $^8/_{12}$.

If I divide each of the third pieces in the first fraction strip into 4 equal parts, then that makes 4 times as many parts that are shaded and 4 times as many parts in all. The 2 shaded parts become $2 \times 4 = 8$ smaller parts, and the 3 total parts become $3 \times 4 = 12$ total smaller parts. So the shaded amount is $^2/_3$ of the strip, but it is also $^8/_{12}$ of the strip:



Note. Adapted from Example 19 on page 42 of the practice guide.

3. Help students make sense of algebraic notation.

Instructional strategies from the examples

- Introduce symbolic notation early and at a moderate pace, allowing students enough time to become familiar and comfortable with it.
- Ask students to explain each component of an algebraic equation by having them link the equation back to the problem they are solving.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2d
TEACHERS: INST.TCK.2, PLAN.SW.3

To allow students time to become comfortable with the symbolic notation used in algebra, teachers should introduce symbolic notation early and at a rate that is not too slow or too fast for learners. One way of accomplishing this is to provide students with arithmetic problems and then support them in translating the problems into algebraic notation. This approach will help students connect their existing arithmetic knowledge with new algebraic knowledge.

Example of linking components of an equation to a problem

Problem
Joseph earned money for selling 7 CDs and his old headphones. He sold the headphones for \$10. He made \$40.31. How much did he sell each CD for?
Solution
The teacher writes this equation: $10 + 7x = 40.31$ **TEACHER:** If *x* represents the number of dollars he sold the CD for, what does the 7*x* represent in the problem? What does the 10 represent? What does the 40.31 represent? What does the 10 + 7*x* represent?

Note. Adapted from Example 21 on page 43 of the practice guide.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
Students' explanations are too short and lack clarity and detail. It is difficult for teachers to identify which mathematical concepts they are using.	Teachers can determine what concepts students are likely to use by solving problems before using them in lessons. Teachers can ask students questions about how a problem was solved and how they thought about the problem. They can also ask students to make a sheet of mathematical rules and use those rules when explaining how they approached a specific problem. The sheet should be brief and include only a few key rules.
Students may be confused by mathematical notations used in algebraic equations.	Teachers should use, and encourage students to use, arbitrary variables in order to help students understand that variables play an abstract role in algebraic equations. For example, use *x* or *y* to represent an unknown quantity. Students may confuse variables if they seem related to items in a problem (for example, if *a* represents apples).

Reference: Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16

This document provides a summary of Recommendation 1 from the WWC practice guide *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students.* Full reference at the bottom of last page.

CONTENT: ***Mathematics***

GRADE LEVEL(S): ***6–12***

LEVEL OF EVIDENCE: ***Minimal***

Recommendation

Use solved problems to engage students in analyzing algebraic reasoning and strategies.

Solving algebraic problems requires students to engage in abstract and critical thinking beyond the arithmetic work they experienced previously. In developing algebraic reasoning, students must analyze and process multiple pieces of information to find a solution to a problem. Examining and discussing possible sources of error and the multiple steps of solved problems will allow students to strengthen their algebraic reasoning skills.

How to carry out the recommendation

1. Have students discuss solved problem structures and solutions to make connections among strategies and reasoning.

Instructional strategies from the examples

- Create opportunities to discuss and analyze solved problems by asking students to describe the steps taken in the solved problem and explain the reasoning used.
- Ask students specific questions about the solution strategy, and whether that strategy is logical and mathematically correct.
- Foster extended analysis by asking students to notice and explain different aspects of a problem's structure.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.1c, PS.1d, PS.2d, PS.3a, PS.3b, PS.3d, PS.7b, PS.7c
TEACHERS: INST.MS.2, INST.PIC.2, INST.AM.4, INST.AM.7, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.3

Teachers should provide opportunities for students to examine solved problems through guiding questions. Teachers can have students explain the reasoning and discuss strategies used. They should keep students engaged and adjust guidance to meet the students' needs and the curricular goals. Guiding questions can be verbal or written. Examples of questions to facilitate student discussions of solved problems include the following:

- What were the steps to solve the problem?
- Could fewer steps have been used?
- Is this a strategy that would work in all cases? Why or why not?
- Is there another way to solve the problem?
- Is there a way to make the solution path more clear?
- What are the mathematical ideas connected to the solution path?

Note. Adapted from Example 1.1 on page 5 in the practice guide.

Teachers can deepen students' analysis and discussion by asking them to focus on the structure of the solved problem. Thinking about structure includes having students examine the mathematical features of a given problem as well as any mathematical relationships that might be present in an expression, representation, or equation. Questions to guide analysis and discussion of structure include the following:

- What quantities are present in this problem? Are they discrete or continuous?
- What operations and relationships among the quantities are shown in the problem? Is the problem expressing an equality or inequality?
- This problem uses parentheses. What do they indicate about the problem's structure?

Note. Adapted from Example 1.2 on page 6 in the practice guide.

2. Select solved problems that reflect the lesson's instructional aim, including problems that illustrate common errors.

Instructional strategies from the examples

- Select problems with varying levels of difficulty and arrange them from simplest to most complex applications of the same concept.
- Show multiple examples simultaneously to encourage students to recognize patterns in the solution steps across problems, or show problems one after the other to facilitate more detailed discussion on each problem.
- Use incorrectly solved problems to help students deepen their understanding by analyzing strategic, reasoning, and procedural errors. Contrast different types of errors with a correctly solved problem.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b
TEACHERS: INST.PIC.2, INST.AM.1, INST.TCK.2, PLAN.SW.3

Discussion of solved problems can help achieve a variety of learning goals, so teachers should align solved problems with their lesson objectives. Sources of solved problems include previous student work, publisher-supplied examples, and those that teachers create on their own. Options for including multiple solved problems in a lesson can include:

- Selecting solved problems that apply the same concept but with varying degrees of difficulty, then presenting them from simplest to most complex application.
- Displaying multiple examples side by side to encourage identifying patterns in the solution steps across problems.
- Showing problems individually to encourage deeper discussion of each problem.

Note. Adapted from page 6 in the practice guide.

When presenting solved problems, teachers should include different solution paths as well as examples that contain errors. Once students examine several correctly solved problems, teachers can use incorrectly solved problems to help them identify and build understanding of concepts and solution processes. The following is a sample procedure for introducing incorrectly solved problems:

- Give students correctly solved problems to study and discuss.
- Once students have an understanding of correct strategies and problems, present an incorrectly solved problem.
- Display the incorrectly solved problem by itself or alongside a correct version of the same problem.
- Clearly label that the problem is solved incorrectly.
- Engage in discussion of the error and what steps led to the incorrect answer.

Note. Taken from Example 1.5 on page 9 in the practice guide.

For examples of ways to present and discuss solved problems, as well as how to align with various learning objectives, see pages 7–11 in the practice guide.

Example of parallel correct and incorrect solved problems, completing the square

Show students the correctly and incorrectly solved problems together. Ask students to describe the error (shown in bold text below), and guide students' discussion of why the error occurred.

	Correct Solved Problem	Incorrect Solved Problem: Strategic and Reasoning Error	Incorrect Solved Problem: Procedural Error
Equation $x^2 + 6x = 27$	$x^2 + 6x = 27$ $x^2 + 6x + 9 = 27 + 9$ $(x + 3)^2 = 36$ $x + 3 = \pm 6$ $x + 3 = 6 \quad x + 3 = -6$ $x = 6 - 3 \quad x = -6 - 3$ $x = 3 \quad x = -9$	$x^2 + 6x = 27$ $x^2 + 6x + 9 = 27 + 9$ $(x + 3)^2 = 36$ $\boldsymbol{x + 3 = 6}$ $x = 6 - 3$ $x = 3$	$x^2 + 6x = 27$ $\boldsymbol{x^2 + 6x + 9 = 27}$ $(x + 3)^2 = 27$ $x + 3 = \pm 3\sqrt{3}$ $x = -3 + 3\sqrt{3} \quad x = -3 - 3\sqrt{3}$

Description of Error	N/A	The student did not include the negative square root as a solution.	The student did not add 9 to both sides when completing the square. This means the new equation is not equivalent to the previous equation.
Questions to Guide Discussion of Error	N/A	If a number squared is 36, what could the number be equal to? What properties of numbers and operations can we use to justify each step in the example?	If you add something to one side of the equation, what else do you need to do? Why? What property is this? The original equation tells us how $x^2 + 6x$ and 27 are related. What is that relationship? If 27 and $x^2 + 6x$ equal each other, then what should be the relationship between 27 and $x^2 + 6x + 9$?

Note. Taken from Example 1.7 on page 11 in the practice guide.

3. Use whole-class discussions, small-group work, and independent practice activities to introduce, elaborate on, and practice working with solved problems.

Instructional strategies from the examples

- Introduce solved problems during whole-class instruction as an overview of a solution strategy.
- Create activities for pairs or small groups of students to critically analyze solved problems.
- After a lesson, give students incomplete solved problems and ask students to complete the solutions.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.1d, PS.2d, PS.3a, PS.3b
TEACHERS: INST.PIC.2, INST.AM.4, INST.AM.7, INST.GS.1

Using solved problems in a variety of contexts may lead to improved use of solution strategies. Teachers can use whole-group instruction to provide an overview of the solution strategy in a solved problem. Next, teachers can allow students to engage with the solved problem in pairs or small groups, including incorrectly solved problems to push students toward deeper, more critical analysis of the problem solution. Teachers can follow this pair or small-group work with whole-group discussion to correct misconceptions and ensure that all components of the problem have been scrutinized. Teachers should move from solved problems to incomplete solved problems, and then to independent practice.

Examples of incomplete solved problems

$-x + 7 \geq 9$ $-x \geq 2$ __________	$3(x + 2) + 12 \leq 4(1 - x)$ __________ $3x + 18 \leq 4 - 4x$ $7x \leq -14$ $x \leq -2$	$2(x + 7) - 5(3 - 2x) \geq 7x - 4$ $2x + 14 - 15 + 10x \geq 7x - 4$ __________ $5x \geq -3$ $x \geq -\frac{3}{5}$

Note. Taken from Example 1.10 on page 14 in the practice guide.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
I already use solved problems during whole-class instruction, but I'm not sure students are fully engaged with them.	Ask questions and be sure to include all students in the discussion to motivate them to think critically. Model think-aloud questions (for example, "Will the strategy work for every problem like this?" "Why or why not?" "How would you modify the solution, if you can, to make it clearer to other students?"). See Examples 1.1 and 1.2 in the practice guide. Additionally, use solved problems beyond whole-group settings to be sure they are scrutinized in more meaningful ways. Include solved problems in class assessments to make whole-class work relevant to students. See Examples 1.9, 1.10, and 1.11 in the practice guide.
I do not know where to find solved problems to use in my classroom and do not have time to make new examples for my lessons	Find sample or worked problems in published curricular materials. Use past or current de-identified student work (such as homework, projects, and assessments) as other examples, particularly for unique solution paths or incorrectly solved problems. Share across classrooms to increase your access.
I'm worried that showing students incorrect solved problems will confuse them.	Although students may not be familiar with examining incorrectly worked problems, doing so can help them build important critical-thinking skills. Be sure that students are clearly aware that a problem contains an error, then focus on the steps to understand the process and where it went wrong. Fully discuss each step to prevent confusion and build recognition and understanding of how the error occurred. See Examples 1.5, 1.6, and 1.7 in the practice guide.

Reference: Star, J. R., Foegen, A., Larson, M. R., McCallum, W. G., Porath, J., & Zbiek, R. M. (2019). *Teaching strategies for improving algebra knowledge in middle and high school students* (NCEE 2015-4010). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/20

This document provides a summary of Recommendation 2 from the WWC practice guide *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students.* Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **6–12**

LEVEL OF EVIDENCE: **Minimal**

Recommendation

Teach students to utilize the structure of algebraic representations.

Examining the underlying structure of an algebra problem (the algebraic representation), regardless of how the problem itself is communicated (for example, symbolic, numeric, verbal, or graphic), can help students see similarities among problems, connections, and solution paths. It also leads to development of understanding about algebraic expressions.

How to carry out the recommendation

1. Promote the use of language that reflects mathematical structure.

Instructional strategies from the examples

- Phrase algebra solution steps in precise mathematical language to communicate the logical meaning of a problem's structure, operations, solution steps, and strategies.
- Use precise mathematical language to help students analyze and verbally describe the specific features that make up the structure of algebraic representations.
- Rephrase student solutions and responses using appropriate mathematical language.

South Carolina standards alignment

MATHEMATICS: PS.3a, PS.3b, PS.6c
TEACHERS: INST.PIC.3, INST.PIC.4, INST.TCK.2, PLAN.SW.2

Using precise language is important for helping students understand algebraic structure. Teachers should use and model precise mathematical language that is related to the structure of the algebraic expression. (See Example 2.2 on page 18 in the practice guide for an example using the distributive property.) Doing so will not only help students develop an understanding of the structure but also lay a foundation for them to reflect, ask questions, and create appropriate representations. Teachers should restate students' responses, using the appropriate mathematical language, to help them grow in their ability to use precise language. Guiding students to use more precise mathematical language helps them focus on and build understanding of the mathematical validity of a problem.

Examples of imprecise language with more precise restatements

Imprecise Language	Precise Mathematical Language
Take out the x.	Factor x from the expression. Divide both sides of the equation by x, with a caution about the possibility of dividing by 0.
Move the 5 over.	Subtract 5 from both sides of the equation.
Use the rainbow method. Use FOIL.	Use the distributive property.

Note. Taken from Example 2.3 on page 18 in the practice guide.

2. Encourage students to use reflective questioning to notice structure as they solve problems.

Instructional strategies from the examples

- Model reflective questioning to students by thinking aloud while solving a problem.
- Share a list of common questions students can ask themselves while s olving a problem.

South Carolina standards alignment

MATHEMATICS: PS.1c, PS.1d, PS.3a, PS.3d, PS.7b, PS.7c
TEACHERS: INST.MS.1, INST.MS.2, INST.AM.4, INST.AM.5, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1

When students ask themselves questions about solving a problem, they are more likely to think about the structure of the problem and solution methods they might use. Teachers should model reflective questions using think-alouds focused on algebraic structure when demonstrating problem-solving. Additionally, providing lists of reflective questions can be helpful as students move from modeling to more independent work. Teachers should encourage students to work in pairs to develop and record their own reflective questions and then carry this practice to independent work.

Examples of reflective questions

- What am I being asked to do in this problem?
- How would I describe this problem using precise mathematical language?
- Is this problem structured similarly to another problem I've seen before?
- How many variables are there?
- What am I trying to solve for?
- What are the relationships between the quantities in this expression or equation?
- How will the placement of the quantities and the operations impact what I do first?

Note. Taken from Example 2.5 on page 20 in the practice guide.

3. Teach students that different algebraic representations can convey different information about an algebra problem.

Instructional strategies from the examples

- Present equations in different forms and ask students to identify similarities and differences.
- Help students see that different representations based on the same information can display the information differently.
- Incorporate diagrams into instruction to demonstrate similarities and differences between representations of algebra problems.

South Carolina standards alignment

MATHEMATICS: PS.1c, PS.2b, PS.2d, PS.4b, PS.7b
TEACHERS: INST.PIC.2, INST.AM.4, INST.TCK.2

Teachers should encourage students to identify and explain different representations of the same problem in order to help them better understand the underlying mathematical structure. During whole-class instruction, teachers should provide a model of both the similarities and differences, showing how different representations of the same information might make solving problems easier. Teachers should also encourage students to see how some representations might better present information about the structure of the problem than others might. As needed, diagrams can help students visualize the problem structure, organize their thoughts about how to solve the problem, and transform the problem into another representation.

Example of using different representations to understand the structure of a problem

Word problem

Ray and Juan both have community garden plots. Ray has a rectangular plot. Juan has a square plot, and each side of his plot is x yards wide. Ray and Juan's plots share one full border; the length of Ray's plot on an unshared side is 4 yards. If Juan and Ray put in a fence around both of their plots, the area of the fenced space would be 21 square yards. How wide is the shared border?

The statement of the problem is one representation of a relationship among three quantities, which are the total area of 21 square yards, the area of Ray's plot, and the area of Juan's plot. Students typically move to other representations to solve the problem. They might draw a diagram and produce an equation, and then solve the equation algebraically or graphically.

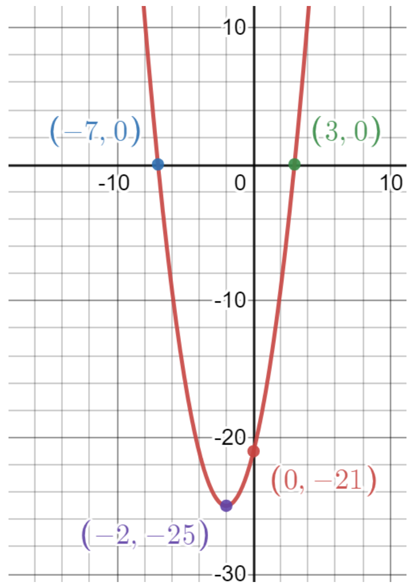
Diagram

The diagram represents the two garden plots with a common border and a 4-yard unshared side of Ray's plot. The diagram also represents one large rectangle composed of two rectangles to illustrate that the total area is equal to the area of Ray's plot plus the area of Juan's plot. Using the rectangles, the given lengths, and the total area of 21 square yards, students can produce and solve an equation.

Students can use the diagram to see the structure of the problem as the equivalence of a total area to the sum of two parts and to express it as an equation. After solving the equation for x, students can explain why there are two possible solutions for the quadratic equation, and why -7 doesn't yield an answer to the question in the word problem.

Equation
Equation representing the equivalent areas in square yards: $\mathbf{21} = \boldsymbol{x}(\mathbf{4} + \boldsymbol{x})$ Equation in standard form: $\mathbf{0} = \boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} - \mathbf{21}$ Equation in factored form: $\mathbf{0} = (\boldsymbol{x} + \mathbf{7})(\boldsymbol{x} - \mathbf{3})$ Total area = 21 yd^2 Area = length × width Area $= \boldsymbol{x}(\mathbf{4} + \boldsymbol{x})$ $\mathbf{21} = \boldsymbol{x}(\mathbf{4} + \boldsymbol{x})$ $\mathbf{21} = \mathbf{4}\boldsymbol{x} + \boldsymbol{x}^{\mathbf{2}}$ $\mathbf{0} = \boldsymbol{x}^{\mathbf{2}} + \mathbf{4}\boldsymbol{x} - \mathbf{21}$ $\mathbf{0} = (\boldsymbol{x} + \mathbf{7})(\boldsymbol{x} - \mathbf{3})$ $\boldsymbol{x} = -\mathbf{7}\ \boldsymbol{x} = \mathbf{3}$ *Students will likely come to the standard form first when solving this problem, then will need to factor to reach the possible solutions for x.* *Students should recognize that the quadratic expression can be factored. The values of x that make the factored expression on the right side of the equation equal to zero can be read from the structure of the expression as a product. For a product to be zero, one of the factors has to be zero, so x is -7 or 3.*

Graph



Students can find where an expression equals zero by thinking of the expression as a function, graphing it, and seeing where the graph crosses the x-axis.

The x-intercepts of the parabola can be read from the factored form. The y-intercept can be read from the standard form, and that form is helpful in determining the vertex of the parabola.

The graph is a parabola because it is a quadratic equation, and the direction in which the parabola opens depends on the sign of the coefficient of x^2.

Note. Adapted from Example 2.8 on pages 22–23 in the practice guide.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
I like to simplify mathematical language, and my students seem to respond positively to my use of simplified and informal language as well. Doesn't this approach make it easier for students than using complicated mathematical language?	Using precise mathematical language ensures that students understand the algebraic concepts in the problem. Language can be simplified but should still clearly link to mathematical structure and ideas. When students use informal language, teachers should restate in correct mathematical language to build students' capacity to use precise language.
My students race through problems. How do I get students to slow down, pause to ask themselves questions, and think about the problem?	Using more challenging or less familiar problems may slow students down and require them to pay attention and notice structure. Another possibility is to have students answer reflective questions as they work or solve problems using multiple solution paths or representations.
Diagrams don't seem to be very useful to some of my students.	Students may not need diagrams to solve problems, but diagrams can help them notice and understand structural components. Make this explicit to students, and continue to model the use of diagrams.

References: Star, J. R., Foegen, A., Larson, M. R., McCallum, W. G., Porath, J., & Zbiek, R. M. (2019). *Teaching strategies for improving algebra knowledge in middle and high school students* (NCEE 2015-4010). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/20

This document provides a summary of Recommendation 3 from the WWC practice guide *Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students.* Full reference at the bottom of last page.

CONTENT: ***Mathematics***

GRADE LEVEL(S): ***6–12***

LEVEL OF EVIDENCE: ***Moderate***

Recommendation

Teach students to intentionally choose from alternative algebraic strategies when solving problems.

Students benefit from learning multiple algebraic strategies to bring to bear on problem-solving. Strategies are more general and abstract than memorized algorithms. Using strategies both requires students to have and provides them with more flexibility in solving problems. Students should not be expected to memorize all possible strategies but have access to multiple strategies for a given problem.

How to carry out the recommendation

1. Teach students to recognize and generate strategies for solving problems.

Instructional strategies from the examples

- Provide students with examples that illustrate the use of multiple algebraic strategies, including standard, commonly used strategies, as well as alternative strategies that may be less obvious.
- Use solved problems that demonstrate how the same problem could be solved with different strategies, as well as how different problems could be solved with the same strategy.
- After students find a solution to a problem, challenge them to solve the problem another way.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.3a, PS.7b
TEACHERS: INST.MS.2, INST.PIC.2, INST.AM.4, INST.AM.9, INST.TCK.2, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1

Teachers can provide well-known, as well as lesser-known, strategies for approaching algebraic problems so students can observe which are effective and efficient in various cases. Teachers should provide solved problems to demonstrate multiple strategies for a single problem as well as strategies that are effective across multiple problems. Doing so will reinforce flexibility of strategy use. In both whole-class instruction and partner work, students should discuss and communicate why a particular strategy is useful.

Examples using different solution strategies

<table>
<tr><th>Conventional Solution Method</th><th colspan="2">Alternative Solution Method(s)</th></tr>
<tr><td colspan="3">Evaluate $\boldsymbol{2a + 4b - 7a + 2b - 8a}$ if $\boldsymbol{a = 1}$ and $\boldsymbol{b = 7}$.</td></tr>
<tr><td>$\boldsymbol{2a + 4b - 7a + 2b - 8a}$
$\boldsymbol{2(1) + 4(7) - 7(1) + 2(7) - 8(1)}$
$\boldsymbol{2 + 28 - 7 + 14 - 8}$
$\boldsymbol{29}$</td><td colspan="2">$2a + 4b - 7a + 2b - 8a$
$-13a + 6b$
$-13(1) + 6(7)$
$-13 + 42$
29</td></tr>
<tr><td colspan="3">Our restaurant bill, including tax but before tip, was $16.00. If we wanted to leave exactly 15% tip, how much money should we leave in total?</td></tr>
<tr><td>$\boldsymbol{16.00 * 1.15 = x}$

$\boldsymbol{x = \$18.40}$</td><td colspan="2">10% of $16.00 is $1.60, and half of $1.60 is $0.80, which totals $2.40, so the total bill with tip would be $16.00 + $2.40 or $18.40.</td></tr>
<tr><td colspan="3">Solve for $\boldsymbol{x}$: $\boldsymbol{3(x + 1) = 15}$</td></tr>
<tr><td>$\boldsymbol{3(x + 1) = 15}$
$\boldsymbol{3x + 3 = 15}$
$\boldsymbol{3x = 12}$
$\boldsymbol{x = 4}$</td><td>$3(x + 1) = 15$
$x + 1 = 5$
$x = 4$</td><td>I know that $3 \times 5 = 15$, so $x + 1$ has to equal 5. That means $x = 4$.</td></tr>
</table>

Note. Adapted from Example 3.1 on page 28 in the practice.

Examples of two different solution strategies to solve the same problem

Strategy 1: Devon's Solution—Apply Distributive Property First	
Solution steps	Labeled steps
$10(y+2) = 6(y+2) + 16$	Distribute
$10y + 20 = 6y + 12 + 16$	Combine like terms
$10y + 20 = 6y + 28$	Subtract $6y$ from both sides
$4y + 20 = 28$	Subtract 20 from both sides
$4y = 8$	Divide by 4 on both sides
$y = 2$	
Strategy 2: Elena's Solution—Collect Like Terms First	
Solution steps	Labeled steps
$10(y+2) = 6(y+2) + 16$	Subtract $6(y = 2)$ on both sides
$4(y+2) = 16$	Divide by 4 on both sides
$y + 2 = 4$	Subtract 2 from both sides
$y = 2$	

Prompts to Accompany the Comparison of Problems, Strategies, and Solutions

- What similarities do you notice? What differences do you notice?
- To solve this problem, what did each person do first? Is that valid mathematically? Was that useful in this problem?
- What connections do you see between the two examples?
- How was Devon reasoning through the problem? How was Elena reasoning through the problem?
- What were they doing differently? How was their reasoning similar? Did they both get the correct solution?
- Will Devon's strategy always work? What about Elena's? Is there another reasonable strategy?
- Which strategy do you prefer? Why?

Note. Taken from Example 3.2 on page 29 in the practice guide.

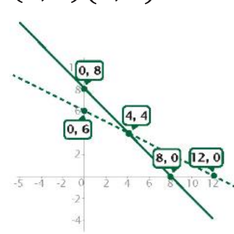
Teachers should introduce one or two strategies at a time to allow students to process new information. They should then work with students to determine which strategies are most effective and efficient through reflective questions that teachers provide or students develop themselves. Students should begin by examining solved problems and then discuss and select strategies for solving other problems during group and individual work. Finally, after students solve a problem, teachers can challenge them to use a different strategy to solve it.

Examples of reflective questions for selecting and considering solution strategies

- What strategies could I use to solve this problem? How many possible strategies are there?
- Of the strategies I know, which seem to best fit this particular problem? Why?
- Is there anything special about this problem that suggests that a particular strategy is or is not applicable or a good idea?
- Why did I choose this strategy to solve this problem?
- Could I use another strategy to check my answer? Is that strategy sufficiently different from the one I originally used?

Note. Taken from Example 3.4 on page 30 in the practice guide.

Examples of possible strategies for solving linear systems

Problem Statement	Solution Strategy	Solution Steps	Notes About Strategies
$5x + 10y = 60$ $x + y = 8$	Graph using *x*- and *y*-intercepts	$5x + 10y = 60$ $(12, 0)(0, 6)$ $x + y = 8$ $(8, 0)(0, 8)$  	The *x*- and *y*-intercepts are integers and easy to find in these two equations, so graphing by hand to find the point of intersection might be a good strategy to use.

Problem Statement	Solution Strategy	Solution Steps	Notes About Strategies
$-2x + y = 7$ $x = 6y + 2$	Substitution	$-2x + y = 7$ $\underline{x = 6y + 2}$ $-2(6y + 2) + y = 7$ $-12y - 4 + y = 7$ $-11y = 11$ $y = -1$	Because one of the equations in this system is already written in the form of *x* =, it makes sense to use the substitution strategy.
$2x + y = 6$ $x - y = 9$	Elimination	$2x + y = 6$ $\underline{x - y = 9}$ $3x = 15$ $x = 5$ $2(5) + y = 6$ $y = -4$	Because the coefficients of the *y* terms are equal in absolute value but have opposite signs, the strategy of elimination may be a natural fit for this system.
$y = 100 + 4x$ $y = 25 + 7x$	Properties of equality	$y = 100 + 4x$ $\underline{y = 25 + 7x}$ $100 + 4x = 25 + 7x$ $75 = 3x$ $x = 25$	Since both equations are in the form of *y* =, it would be logical to set the two expressions in *x* equal to each other and solve for *x*.

Note. Adapted from Example 3.6 on page 32 in the practice guide.

2. Encourage students to articulate the reasoning behind their choice of strategy and the mathematical validity of their strategy when solving problems.

Instructional strategies from the examples

- Have students describe their reasoning while analyzing the problem structure, determining their solution strategy, solving a problem, and/or analyzing another student's solution.
- When introducing group activities, model how to work with a partner to discuss potential strategies, how to label the steps of each strategy, and how to explain the similarities and differences observed between strategies.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.1c, PS.1d, PS.3a, PS.6c, PS.7c
TEACHERS: INST.AM.4, INST.AM.7, INST.TCK.2, INST.PS.1, PLAN.SW.1, PLAN.SW.2, PLAN.Desc.1

To help students better understand their choices and goals in problem-solving, teachers should provide multiple opportunities to analyze problem structures, determine solution strategies, describe reasoning while solving problems, and analyze other students' solution strategies. Students should demonstrate their strategic thinking for each step of the solution process, both verbally and in writing. Teachers can provide a model as well as a list of guiding questions, such as "What do you notice about the structure of this problem?" and "How does that point you toward a particular strategy to solve it?"

Example prompts to encourage students to articulate their reasoning

- What did you notice first about the problem structure? How did that influence your solution strategy? What strategy is appropriate for solving this problem and why?
- What choices did you have to make in solving this problem?
- What goal were you trying to achieve?
- How did you get your answer? How do you know it is correct?
- Describe to another student how to solve this problem.
- What was most difficult about this problem? Did you run into any challenges? If so, what did you do to overcome them?

Note. Taken from Example 3.7 on page 33 in the practice guide.

3. Have students evaluate and compare different strategies for solving problems.

Instructional strategies from the examples

- Have students compare problem structures and solution strategies to discover relationships among similar and different problems, strategies, and/or solutions.
- Use solved problems showing two strategies side by side to enable students to see the number, type, and sequence of solution steps.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.1c, PS.1d, PS.3b, PS.3d, PS.7b, PS.7c
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.5, INST.AM.7, INST.AM.9, INST.TH.2, INST.PS.1, PLAN.SW.1, PLAN.Desc.1

Once students have mastered a strategy, teachers should have them make comparisons across similar and different problem structures and strategies to identify relationships. Teachers should support students in considering how a solution strategy is similar to and different from others they have encountered. Teachers could encourage students to think about the accuracy, efficiency, and applicability of various problem-solving strategies. Guided discussion using worked problems can be helpful as students move from teacher-mediated to more individual work.

Example of small-group comparison and discussion activity

Objectives:

☑ Share and compare multiple solution strategies

☑ Use precise mathematical language to describe solution steps

☑ Explain reasoning and mathematical validity

Directions: Pair students off to work on algebra problems so that students with different strategies have the opportunity to talk with each other. For example, if two strategies are prevalent and approximately half of the students use each, students may be put into groups A and B based on like strategies and then each paired with a student from the other group. Partners can discuss the strategies they used to solve the first problem (e.g., What strategy did each person use? How did the strategies differ from one another? What was the partner's rationale for using a different strategy? Did both strategies produce the same answer?). Challenge students to use their partner's strategy when solving the next problem. Conclude the activity by asking students to reflect on what they discussed with their partners, explaining the most important ways in which the two strategies differ. Have students record the strategies discussed by the class.

Note. Taken from Example 3.9 on page 34 in the practice guide.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
I'm worried about confusing my students by teaching them multiple strategies for solving a problem. They have a hard enough time learning one strategy! Isn't it easier for them to master one strategy for solving algebra problems?	Students are not expected to become experts in all strategies but to clarify their thinking when choosing the most appropriate one for a given problem. Teachers can focus on teaching one strategy at a time and then ask students to compare a new strategy with an established one. Different students may be more comfortable with certain strategies, so allowing them to explore multiple strategies will be helpful.
My special education students need a very structured process for solving algebra problems. Introducing multiple strategies and asking students to choose among strategies might be hard on them.	Teachers can provide explicit instruction to students with disabilities while still teaching them alternative strategies. Instruction should include both the steps and a clear rationale for application. Asking students to simply memorize a single strategy without building their understanding of how and why it is appropriate for a given problem type will lead to challenges for students in special education.
I can't seem to teach my students multiple strategies without them becoming confused. I presented and compared five algebra strategies for solving quadratic equations during one class period, and my students didn't understand. What should I do?	Students may not need diagrams to solve problems, but they will need them to notice and understand structural components. Make this explicit to students and continue to model the use of diagrams.

Roadblock	Suggested Approach
Teaching students to use and compare multiple strategies requires knowledge of many strategies and our textbook presents only one strategy.	The full practice guide provides lists of strategies for quadratic and linear system equations. Teachers can share strategies with one another in professional learning communities and create class posters or handouts for their students to access.
How can I stay on schedule teaching everything required to meet state standards and still have time to teach students to use multiple strategies?	Teachers can incorporate alternative and multiple strategies into existing lessons to develop students' critical thinking and algebraic reasoning. The focus should be on helping students reason algebraically and recognize when an alternative strategy might provide a solution that is more effective or efficient.

Reference: Star, J. R., Foegen, A., Larson, M. R., McCallum, W. G., Porath, J., & Zbiek, R. M. (2019). *Teaching strategies for improving algebra knowledge in middle and high school students* (NCEE 2015-4010). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/20

This document provides a summary of Recommendation 1 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten through 8th Grade*. Full reference at the bottom of the last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–2**

LEVEL OF EVIDENCE: **Minimal**

Recommendation

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

Students develop a basic understanding of fractions before starting kindergarten through activities such as equally sharing a set of objects with a group of people and proportional reasoning. Teachers can build on this knowledge as they introduce students to the more formal concept of fractions (either as division or ratios), building on what students already understand about sharing equally. Although fraction concepts are usually not introduced until first or second grade, sharing activities can begin as early as pre-K, setting the stage for a deeper understanding of fractions.

Equal sharing	By age 4, students can distribute equal numbers of equal-size objects among a small number of recipients, and the ability to equally share improves with age. Sharing a set of discrete objects (e.g., 12 grapes shared among three students) tends to be easier for young students than sharing a single object (e.g., a candy bar), but by age 5 or 6, students are reasonably skilled at both.
Proportional relations	By age 6, students can match equivalent proportions represented by different geometric figures and by everyday objects of different shapes. One-half is an important landmark in comparing proportions; students more often succeed with comparisons in which one proportion is more than half and the other is less than half than on comparisons in which both proportions are more than half, or both are less than half (e.g., comparing $^1/_3$ to $^3/_5$ is easier than comparing $^2/_3$ to $^4/_5$). In addition, students can complete analogies based on

	proportional relations—for example, a half-circle is to a half-rectangle as a quarter-circle is to a quarter-rectangle.

Note. Taken from page 13 of the practice guide.

How to carry out the recommendation

1. Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.

Instructional strategies from the examples

- Begin equal-sharing activities with a set of objects that can be evenly distributed between two people, then move to larger numbers of people.
- Once students have gained facility with equally sharing a group of objects, move to partitioning a single object into fractional parts for equal sharing.
- As teachers increase the number of people to share with, teachers should consider using multiples of two to allow students to use repeated halving to help solve problems. Later, teachers should move to activities that can't be solved with repeated halving.

South Carolina standards alignment

MATHEMATICS: 2.ATO.3
TEACHERS: INST.MS.2, PLAN.SW.3

Teachers should provide students with sharing activities that progressively build on their existing strategies for dividing, beginning with those involving equal sharing of a set of objects (first between two people, then increasing the number of people), then progressing to activities involving partitioning a set of objects, then a single object, into different fractional parts. Teachers should encourage students to use manipulatives (e.g., counters, beans) or drawings when doing these activities. As students work through activities and different representations, teachers can introduce formal fraction names and have students label fractions on their drawings. To most effectively build students' understanding, engage them in a variety of naming and labeling activities.

Sharing a set of objects. In early equal-sharing activities, teachers should have students solve problems that involve two people (then increase the number of people) and a set of objects that can be divided equally with none left over. To further develop students' understanding, teachers should present situations by describing the number of items to be shared and the number of people who will be sharing, then have students work to determine the number of items each person should receive. As students gain success, teachers can pose a similar problem with the same number of items but increase the number of people. Note that problems of this type should always focus on equal sharing among the recipients.

Example of sharing a set of objects evenly among recipients

Problem
Three students want to share 12 cookies so that each student receives the same number of cookies. How many cookies should each student get?
Example of student solution strategy
Students can solve this problem by drawing three figures to represent the students and then drawing cookies by each figure, giving one cookie to the first student, one to the second, and one to the third, continuing until they have distributed 12 cookies to the three students, and then counting the number of cookies distributed to each student. Other students may solve the problem by simply dealing the cookies into three piles as if they were dealing cards. 

Note. Taken from Figure 1 on page 14 of the practice guide.

Partitioning a single object. After students gain facility with equal-sharing activities, teachers should present problems that involve dividing one or more objects into equal parts. Thus, the focus of the problems shifts from *how many* objects each person has, to *how much* of an object each person should get. For initial problems, teachers should present one object to be shared between people, then progress to multiple objects being divided into smaller parts to be shared equally among people.

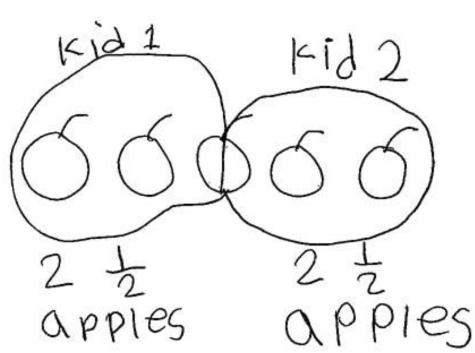
Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Problems involving people equally dividing one object result in unit fractions (e.g., $^1/_2$, $^1/_3$, $^1/_5$), while the more complex problems of multiple objects and multiple people often result in non-unit fractions (e.g., $^1/_2$).

Teachers should also consider sharing among groups that are multiples of 2 (e.g., 2, 4, 8, 16 . . .). This allows students to partition using repeated halving strategies. Later, teachers should introduce problems that cannot be solved with this strategy to help students develop other strategies. For example, teachers may provide students with toothpicks or wooden sticks to lay across objects to represent cutting or partitioning.

Example of partitioning both multiple and single objects

Problem
Two students want to share five apples that are the same size so that both have the same amount to eat. Draw a picture to show what each student should receive.
Example of student solution strategy
Students might solve this problem by drawing five circles to represent the five apples and two figures to represent the two students. Students might then draw lines connecting each student to two apples. Finally, they might draw a line partitioning the final apple into two approximately equal parts and draw a line from each part to the two students. Alternatively, as in the picture below, students might draw a large circle representing each student, two apples within each circle, and a fifth apple straddling the circles representing the two students. In yet another possibility, students might divide each apple into two parts and then connect five half apples to the representation of each figure.



Note. Taken from Figure 2 on page 15 of the practice guide.

2. Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.

Instructional strategies from the examples

- As teachers gradually increase the number of people to share with, guide students to understand the inverse relationship between the number of people and the amount each person receives.
- Provide activities for students that also involve partitioning the number of sharers (e.g., move from one group sharing to the group now being divided equally at different tables and sharing).

South Carolina standards alignment

MATHEMATICS: 3.NSF.1a, 3.NSF.2a, 3.NSF.2b
TEACHERS: PLAN.SW.3

Building on the activities in Step 1 above, teachers can extend students' understanding to ordering fractions and identifying equivalent fractions using similar story problems involving a group of people sharing objects when students continue to use manipulatives and/or drawings to help make sense of the problem. However, teachers extend the scenarios to require fraction comparisons or identification of equivalent fractions by focusing on different aspects of students' solution strategies.

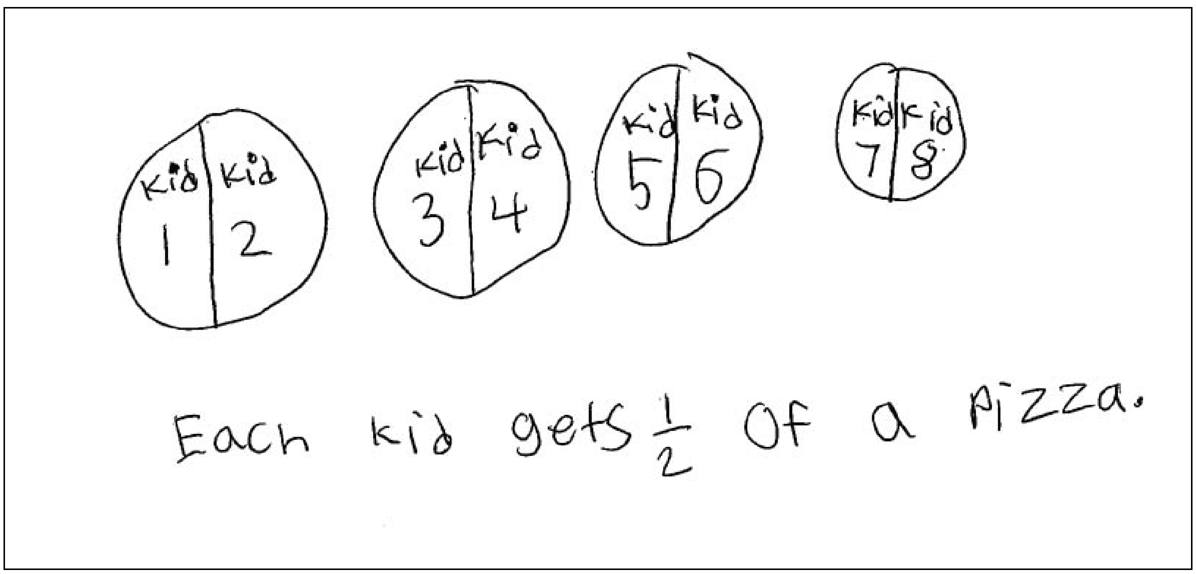
To help students better understand relative size, for example, teachers can gradually increase the number of people who will be sharing and have students compare the relative amounts that each person receives. In this, teachers should guide students to observe that increasing the number of people reduces the amount that each person receives and vice versa. Teachers should help students link this idea to the formal fraction names they have been using to identify the quantities, guiding them to use these names to discuss the results of their solutions (e.g., $^1/_3$ of an object is greater than $^1/_4$ of that same object).

In these scenarios, teachers should include both of the following approaches:

- **Partition objects into larger and smaller pieces.** Teachers should guide students to think about different ways to solve the same problem. This might involve partitioning the object(s) into smaller or larger pieces while ensuring that each person receives an equal amount. In this way, teachers can help students see that groups of the smaller pieces can be combined to be the same size as a larger piece. For example, as shown below, a student may solve the problem of eight people sharing four pizzas by cutting each pizza in half and giving each person half of a pizza. This problem could also be solved by cutting each pizza into quarters and giving each person two quarters. Thus, one half is the same as, or equivalent to, two quarters (i.e., $^1/_2 = {}^2/_4$).

- **Partition the number of sharers and the number of items.** In this approach, teachers help students build the idea of fraction equivalence by partitioning both the number of sharers and the number of objects. For example, in the problem shared previously, a student may partition each pizza into eighths and give each person four pieces or four eighths. The teacher could extend the problem to have the student compare what would happen if the people were split into two tables so that each table of four had to share two pizzas . . . or that people were split into four tables. In each case, each person still receives one-half of the pizza. Every situation is equivalent.

Example of student work for sharing four pizzas among eight students



Note. Taken from Figure 3 on page 16 of the practice guide.

Teachers can also help students develop their understanding of equivalent fractions by using “missing value” problems. Here, students need to determine the number of objects they would need to result in an equivalent share. For example, the practice guide (p. 17) presents the following: “If six students share eight oranges at one table, how many oranges are needed at a table of three students to ensure each student receives the same amount?” (Note: Further scenarios and examples of students’ thinking are provided on p. 17 of the practice guide.)

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

3. Build on students' informal understanding to develop a more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

Instructional strategies from the examples

- Use proportional relations, covariation, and/or patterns to help develop students' proportional thinking.
- Use scenarios that can't directly be quantified to help students think about proportional relations outside of a full focus on numbers.

South Carolina standards alignment

MATHEMATICS: 3.NSF
TEACHERS: PLAN.SW.3

To help build students' understanding of proportional reasoning, teachers might use a story like *Goldilocks and the Three Bears* that asks students to think about the proportional relationships between pairs of objects without providing specific numbers (e.g., Papa Bear goes with the large chair, Baby Bear goes with the small bed). Here are some examples of different relations teachers might use:

- **Proportional relations.** As with *Goldilocks and the Three Bears*, teachers present students with basic proportional relations that are not quantified (e.g., do not present actual values). For example, how many students would it take to balance a seesaw with one, two, or three adults on the other side?
- **Covariations.** These types of scenarios involve one quantity increasing while another is also increasing (e.g., the age of a student and the height of the student).
- **Patterns.** Teachers can use simple repeating patterns to develop a scenario (e.g., RRYRRYRRYRRY), then discuss with students how many red beads there are for every yellow bead. Students can then change the pattern to a different ratio or extend the pattern to solidify proportional reasoning.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
Students are unable to draw equal-sized parts.	Teachers should let students know that it's okay to draw parts that aren't exactly equal as long as they recognize that the parts are supposed to be equal when they work on solving the problem.
Students do not share all the items (non-exhaustive sharing) or do not create equal shares.	Although teachers are building on students' intuitive understanding of sharing equally, students will still make mistakes, such as not sharing all the objects, especially when the problem involves partitioning one or more of the objects. Teachers should focus on helping students understand that sharing involves using all the objects (e.g., letting students know that each person needs to get all they possibly can). If students do not create equal shares, teachers can reframe by reminding students that each person needs to receive the same amount (e.g., things need to be shared fairly). Equal sharing like this helps lay the foundation for students' understanding of equivalent fractions and equivalent magnitude difference (e.g., that a difference of ½ is the same whether it's between 0 and ½ or 23 and 23 ½).
When creating equal shares, students do not distinguish between the number of things shared and the quantity shared.	Limited experience often leads students to make the mistake of giving each person the same number of items rather than thinking about the amount, especially when items shared may be of different sizes. Teachers can use color cues to help students distinguish between the number and the amount. For example, if five small and five large food items are being shared between two dogs, the teacher may color the large items green and the small items red, give all five green items to one dog and all five red items to the other, then ask students if the dogs received the same amount of food.

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15

This document provides a summary of Recommendation 2 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Initial fraction learning often starts with the concept of fractions as part of a whole, but that understanding does not express the fact that fractions are numbers with magnitudes that can be compared (e.g., ordered, considered equivalent). Missing this understanding is often at the root of many misconceptions about fractions (e.g., adding to fractions by adding the numerators, then adding the denominators; not seeing fractions as numbers or units of measurement). Using number lines is an effective way to help develop an understanding of fractions as numbers with magnitudes because they can provide a clear picture of the magnitude of fractions, the relationship between fractions and whole numbers, and the relationship between fractions, decimals, and percentages. Number lines also provide a foundation for students' number sense with fractions and a way to visualize negative fractions.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

How to carry out the recommendation

1. Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.

Instructional strategies from the examples

- Use measurement activities to develop the idea that fractions allow for more precise measurement than whole numbers.
- Present situations in which fractions are used to solve problems that cannot be solved only using whole numbers.
- Show students the various measurement lines on a measuring cup and convey the importance of fractions in describing quantities.

South Carolina standards alignment

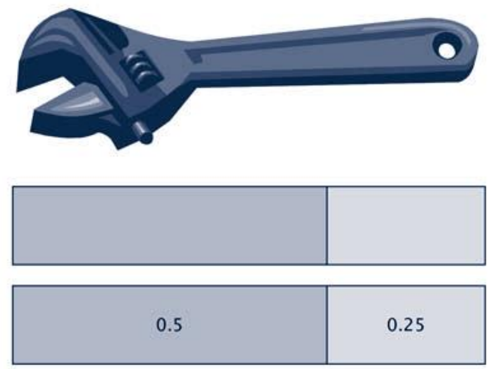
MATHEMATICS: PS.1a, PS.2a, PS.2b
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

Helping students view fractions as numbers opens the door for them to use fractions to measure quantities and understand that using fractions allows for more precise measurement of objects. Fractions can also help solve certain problems that cannot be solved using only whole numbers (e.g., describing the amount of sugar needed for a recipe that asks for more than 1 cup but less than 2 cups). Fraction strips are length models that teachers can use to reinforce the idea of fractions representing quantities. These fraction strips are also known as fraction strip drawings, strip diagrams, bar strip diagrams, and tape diagrams.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Example of measurement activities with fraction strips

Using fraction strips to measure an object



Teachers can use fraction strips as the basis for measurement activities to reinforce the concept that fractions are numbers that represent quantities.

To start, students can take a strip of card stock or construction paper that represents the initial unit of measure (i.e., a whole) and use that strip to measure objects in the classroom (desk, chalkboard, book, etc.). When the length of an object is not equal to a whole number of strips, teachers can provide students with strips that represent fractional amounts of the original strip. For example, a student might use three whole strips and a half strip to measure a desk.

Teachers should emphasize that fraction strips represent different units of measure and should have students measure the same object first using only whole strips and then using a fractional strip. Teachers should discuss how the length of the object remains the same but how different units of measure allow for better precision in describing it. Students should realize that the size of the subsequently presented fraction strip is defined by the size of the original strip (i.e., a half strip is equal to one half of the length of the original strip).

Note. Taken from Example 1 on page 21 in the practice guide.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

2. Provide opportunities for students to locate and compare fractions on number lines.

Instructional strategies from the examples

- Provide opportunities for students to locate and compare fractions on number lines, beginning with fully labeled number lines and gradually reducing the labels.
- Use number lines to compare various fractions to whole numbers greater than one.
- Encourage students to think about the distance between two fractions.

South Carolina standards alignment

MATHEMATICS: PS.2a, PS.2b
TEACHERS: INST.PIC.2

Activities involving comparing fractions should include a variety of fraction forms, such as proper fractions, improper fractions, mixed numbers, whole numbers, decimals, and percentages. Teachers should start by providing number lines with fractional parts already marked and move to numbers lines that are more minimally labeled. This helps avoid student errors in partitioning accurately and also allows for location and comparison of fractions whose locations are indicated (e.g., $^3/_8$ and $^5/_8$ on a ruler), as well as fractions whose denominators are a factor of the indicated fractions (e.g., $^1/_4$ and $^3/_4$) and fractions between those indicated (e.g., $^1/_7$ and $^3/_5$).

Activities should also involve comparison with whole numbers, include fractions equivalent to whole numbers (e.g., locating 1 and $^8/_8$) and comparing fractions of various sizes to whole numbers greater than one (e.g., locating $^{10}/_3$ on a number line, with 0 at the left end and 5 at the right end). To assist students in comparing fractions with different denominators, teachers can label number lines with one fractional-unit sequence above and a different sequence below.

One activity teachers can use with the whole class involves drawing a number line on the board and having students estimate and mark where different fractions fall. The teacher can guide students' discussion about fractions yet to be placed, focusing on maintaining the correct order and encouraging thinking about the distance between fractions. Again, including a variety of fraction types, and also decimals and percentages, helps solidify student understanding.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Example of introducing fractions on a number line

To illustrate the location of $^3/_5$ on a 0-to-5 number line, a teacher might first mark and label the location of 1 and then divide the space between each whole number into five equal-size parts. After this, they might add the labels $^0/_5$, $^1/_5$, $^2/_5$, $^3/_5$, $^4/_5$, and $^5/_5$ in the 0–1 portion of the number line and highlight the location of $^3/_5$. Displaying whole numbers as fractions (e.g., $^5/_5$) allows teachers to discuss what it means to describe whole numbers in terms of fractions and to clarify that whole numbers are fractions, too.

Note. Adapted from Example 2 on page 22 in the practice guide.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

3. Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.

Instructional strategies from the examples

- Use number lines to:
  - Illustrate that equivalent fractions describe the same magnitude.
  - Illustrate that an infinite number of fractions exist between any two other fractions.
  - Introduce negative fractions.
  - Convey the symmetry of positive and negative fractions about zero.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, PS.2b
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

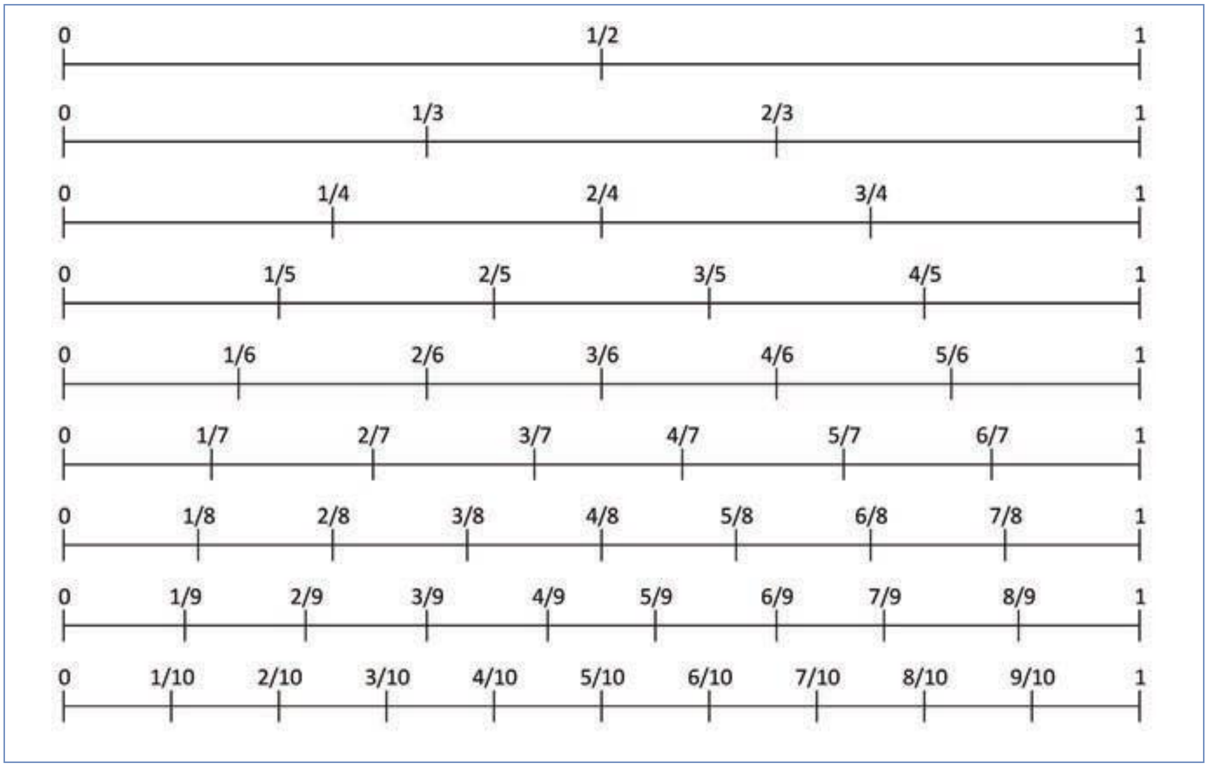
Number lines are also useful tools for helping students think about and identify equivalent fractions, negative fractions, and fraction density.

To illustrate that equivalent fractions have the same magnitude, a teacher might provide a number line labeled with one fractional sequence on top and a different fractional sequence on the bottom. Students can locate fractions on each of the sequences and discuss how they are equivalent when they are in the same location on the number line. Discussions about equivalent fractions should build the measurement activities from Step 1 above, using rulers and fraction strips to reinforce.

Another challenging concept is for students to understand that an infinite number of fractions exist between any two other fractions on the number line. This is an important difference between fractions and whole numbers. To help students understand this concept, teachers can ask them to successively partition a portion of the number line into smaller and smaller unit fractions (e.g., repeatedly dividing a whole number segment in half). Similarly, teachers should use decimals and percentages to represent this concept.

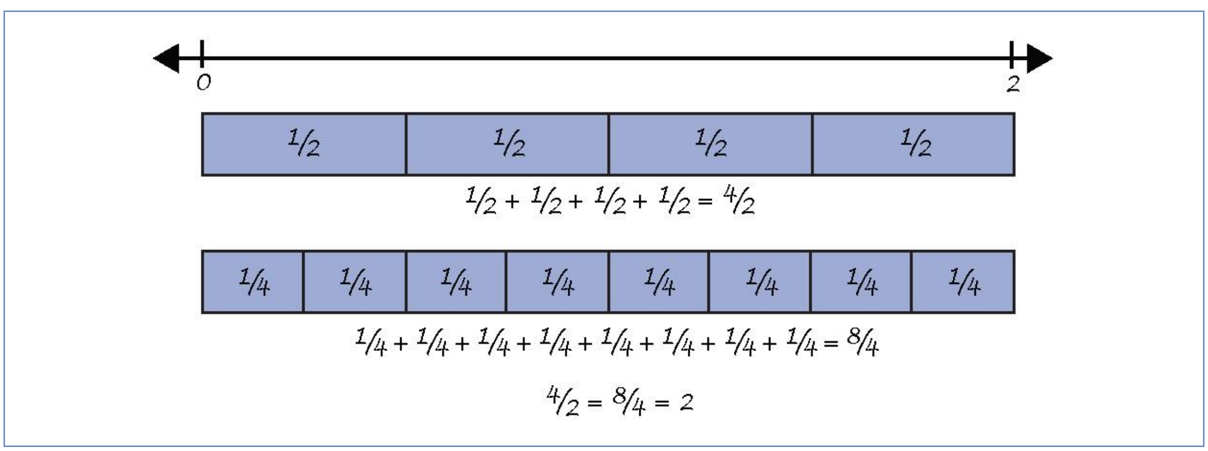
The number line provides an excellent visual representation of fractions both greater than and less than zero. By providing number lines that include marks and labels for zero, several positive fractions, and several negative fractions with the same absolute values as the positive fractions, teachers can help convey the symmetry about zero of positive and negative fractions.

Example of using numbers lines to show equivalence of fractions



Note. Taken from Example 4 on page 23 in the practice guide.

Example of using fraction strips to demonstrate equivalent fractions



Note. Taken from Example 5 on page 24 in the practice guide.

4. Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

Instructional strategies from the examples

- Use a number line with common fractions listed above with equivalent decimals and percentages below.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, PS.2b, PS.6b, 6.NS.1, 6.NS.9, 6.RP.3e, 7.NS.5, 8.NS.3
TEACHERS: INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.3

Number lines can provide a useful tool in helping students develop a broad view of fractions as numbers, including understanding that fractions can be represented as decimals and percentages, as well as common fractions. Fractions, decimals, and percentages are just different ways of representing the same number. By using a number line with common fractions listed above and decimals or percentages below, students can locate and compare fractions, decimals, and percentages on the same number line.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Students try to partition the number line into fourths by drawing four hash marks rather than three, or they treat the whole number line as the unit.	Teachers may need to teach students that "fourths" is represented as four equal segments between two whole numbers, demonstrating that three equally spaced hatch marks divides the space into four equal parts. Later, guide students to generalize this rule to know that to divide a portion of the number line between two whole numbers into $^1/_n$ units involves drawing $n - 1$ hatch marks between the two whole numbers.
When students locate fractions on the number line, they treat the numbers in the fraction as whole numbers (e.g., placing 3/4 between 3 and 4).	This error is based in students' misconception where they try to apply whole number understanding to fractions. Teachers can help address this misconception through using number lines and providing contrasting cases where students locate, for example, 3 and 4 on a 0-to-4 number line, then locate $^3/_4$ between 0 and 1. Teachers should follow this activity with discussion about why each is placed where it is on the number line.
Students have difficulty understanding that two equivalent fractions are the same point on a number line.	Teachers can show students one set of labels above the number line and another below the number line. For example, halves could be marked above and eighths below. Teachers can then point to the equivalent positions of $^1/_2$ and $^4/_8$ or 1, $^2/_2$, and $^8/_8$, and so on. Another approach would be for students to create a number line showing $^1/_2$ and another number line showing $^4/_8$, and then compare the two. Teachers can line up the two number lines and lead a discussion about equivalent fractions.

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Roadblock	Suggested Approach
The curriculum materials used by my school district focus on part-whole representations and do not use the number line as a key representational tool for fraction concepts and operations.	Understanding that fractions can represent part of a whole is only one use. Using measurement and number lines provides an additional context to help students build a rich understanding of fractions. Teachers can also use manipulatives designed to represent parts of a whole for measurement purposes; when doing this, teachers should make sure students aren't simply counting pieces represented by the numerator and denominator but are instead seeing the fraction as a single quantity. Using number lines that are unmarked between endpoints can help, because they do not provide objects to count. Additionally, seek out other textbooks and resources that may provide examples of using fractions as measures of quantity.

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15

This document provides a summary of Recommendation 3 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Help students understand why procedures for computations with fractions make sense.

Building conceptual understanding is foundational for students' proficient use of computational procedures, but procedures with fractions are often taught without helping students understand how or why the procedures work. Teachers should help students build both procedural fluency and conceptual understanding by explaining to students how the computations procedures they are presenting transform fractions in meaningful ways. Using practices such as visual representations, estimation, and setting problems in real-world contexts help reinforce students' conceptual understanding.

How to carry out the recommendation

1. Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.

Instructional strategies from the examples

- Use visual representations and manipulatives to build insight into concepts underlying computational procedures and the reasons why these procedures work.
- Model the division of fractions with representations such as ribbons or a number line.
- Consider the advantages and disadvantages of different representations for teaching procedures for computing with fractions and whether a representation can be used with different types of fractions.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.1c, PS.2b, PS.2c, PS.4a, PS.4b, 6.NS.1, 6.RP.3b
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

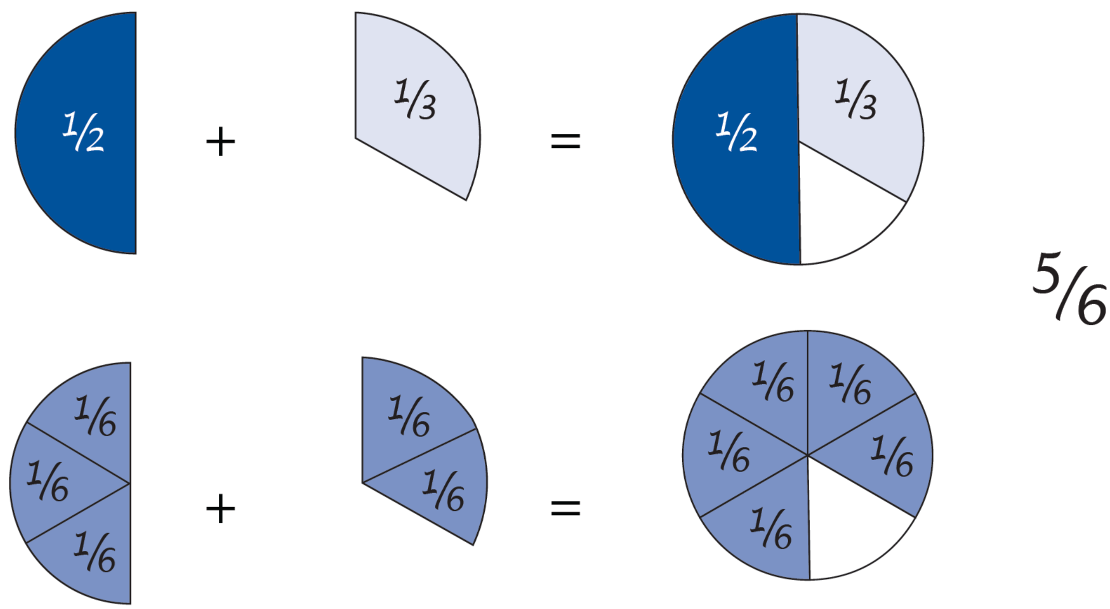
Teachers can use visual representations and manipulatives (e.g., number lines and area models) to help students understand basic concepts underlying conceptual procedures with fractions and the reasons why the procedures work. Here are some examples of how teachers can use representations of fractions in this way:

- **Find a common denominator when adding and subtracting fractions.** A common misunderstanding with students is that you can add fractions by adding the numerators and adding the denominators. Teachers can use a variety of representations to provide visual cues to help students see the need for common denominators. For example, if students are given two circle fractions representing $1/_2$ and $1/_3$, the teacher can show how converting both into sixths provides a common denominator that applies to both fractions, allowing the student to add the two fractions together (see Example 1 below). Teachers should build on this understanding by discussing with students why multiplying the denominators results in a common denominator that applies to both original fractions.
- **Redefine the unit when multiplying fractions.** Using concrete or pictorial representations can help students visualize that multiplying two fractions equates to finding a fraction of a fraction. Area models, for example, provide good examples, as shown in Example 2 below. The approach in Example 2 demonstrates how the initial unit is the full cake, treating the cake as the whole and dividing it into thirds, then moving to treat the portion shaded in the first step as the whole and dividing it into fourths.
- **Divide a number into fractional parts.** Although dividing fractions may look different on the surface, it is conceptually similar to dividing whole numbers. That is, students can think about how many times the divisor goes into the dividend. Teachers can use representations such as number lines and ribbons to model division of fractions. For example, in Example 3 below, ribbons are used to model $1/_2 \div 1/_4$, helping students think in terms of "How many $1/_4$s are there in $1/_2$?"

When selecting representation, teachers should consider the advantages and disadvantages of each to ensure that the representation adequately reflects the computational process students are expected to learn, allowing them to make connections between the representation and the computation. Teachers should also think about the facility of the representation in working with different types of fractions (e.g., proper fractions, mixed numbers, improper fractions, and negative fractions), as well as with other mathematical concepts (e.g., representing decimals with base-10 blocks, using 100 grids where one square is seen as $1/_{100}$ of the whole).

Example 1. Using fraction circles for addition

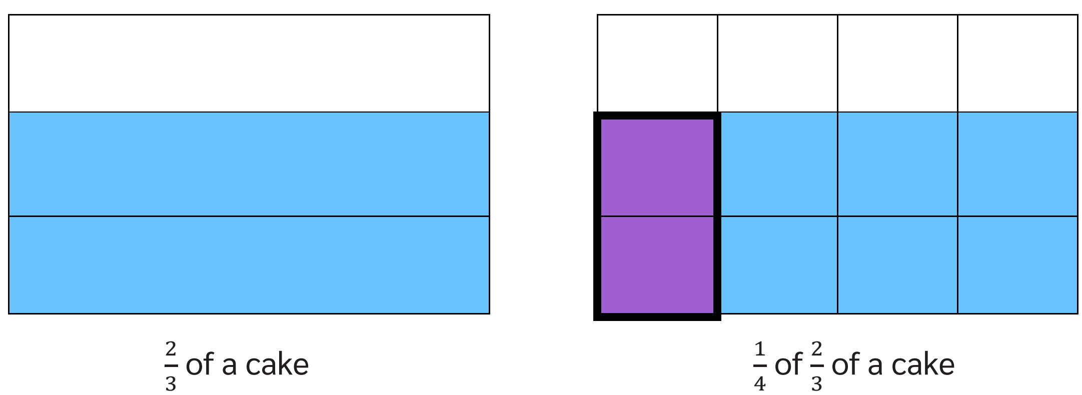
Adding $^1/_2 + {}^1/_3$ using fraction circles:



Note. Taken from Figure 6 on page 28 in the practice guide.

Example 2. Redefining the unit when multiplying fractions

Lori is icing a cake. She knows that 1 cup of icing will cover $^2/_3$ of a cake. How much cake can she cover with $^1/_4$ cup of icing?



$\frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$ of a cake

Note. Taken from Figure 7 on page 29 in the practice guide.

Example 3. Using ribbons to model division with fractions

Students use ribbons to solve $^1/_2 \div {}^1/_4$.

Step 1. Divide a ribbon into fourths.

$^1/_4$	$^1/_4$	$^1/_4$	$^1/_4$

Step 2. Divide a ribbon of the same length into halves.

$^1/_2$	$^1/_2$

Step 3. Compare the two ribbons to find out how many fourths of a ribbon are the same length as (that is, "can fit it into") one half.

$^1/_4$	$^1/_4$
⇩	⇩
$^1/_2$	

Two-fourths are the same length as ("fit into") one half of the ribbon.

So $^1/_2 \div {}^1/_4 = 2.$

Note. Adapted from Figure 8 on page 30 in the practice guide.

2. Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.

Instructional strategies from the examples

- Provide opportunities for students to estimate the solutions to problems.
- Discuss whether and why students' solutions to specific problems are reasonable to help improve students' estimation skills.
- Explicitly teach estimation strategies.

South Carolina standards alignment

MATHEMATICS: PS.1d, PS.2a, PS.4c
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.5, INST.PS.1

In conjunction with teacher computation with fractions, providing students opportunities to estimate their solutions helps build their reasoning skills and focus on the meaning of the computational procedures. Teachers should ask students to provide an initial estimation and explain their thinking before having them actually compute the answer to help students judge the reasonableness of the answers they compute. Teachers can help improve both students' estimation skills and their judgment of the reasonableness of computed answers by having students discuss the strategies they used to determine their estimate and compare their initial estimates to the solutions they've computed.

Teachers should provide opportunities for students to estimate solutions for problems in which the solution cannot be easily computed. Additionally, explicitly teaching effective strategies for estimation can help maximize the value that estimation has on deepening students' understanding of fraction computation.

Example 4. Discussing estimation strategies and reasonableness of solution

Problem
${}^{1}/_{2} + {}^{1}/_{5}$

Solution
Student: I estimate that $^1/_2 + {}^1/_5$ is more than $^1/_2$ but less than $^3/_4$.   **Teacher:** Why do you think that?   **Student:** Well, I know that $^1/_2 + {}^1/_4 = {}^3/_4$. $^1/_5$ is less than $^1/_4$, so the answer for $^1/_2 + {}^1/_5$ will be less than $^3/_4$.   **Teacher:** So, let's compute the sum and see.   *Student computes the sum and arrives at a solution of $^2/_7$ because they incorrectly add the numerators and denominators.*   **Teacher:** Do you think that $^2/_7$ is a reasonable solution for $^1/_2 + {}^1/_5$?   **Student:** It can't be, because that's less than $^1/_2$, and I know the answer has to be bigger than $^1/_2$.   *Teacher follows up with further discussion of the procedure the student used, helping them identify the error made and guiding them to understand the correct procedure and solution.*

Note. Adapted from example on page 31 in the practice guide.

Example 5. Strategies for estimating with fractions

Strengthening estimation skills can develop students' understanding of computational procedures.

Benchmarks. One way to estimate is through benchmarks—numbers that serve as reference points for estimating the value of a fraction. The numbers 0, $^1/_2$, and 1 are useful benchmarks because students generally feel comfortable with them. Students can consider whether a fraction is closest to 0, $^1/_2$, or 1. For example, when adding $^7/_8$ and $^3/_7$, students may reason that $^7/_8$ is close to 1, and $^3/_7$ is close to $^1/_2$, so the answer will be close to $1^1/_2$. Further, if dividing 5 by $^5/_6$, students might reason that $^5/_6$ is close to 1, and 5 divided by 1 is 5, so the solution must be a little more than 5.

Relative Size of Unit Fractions. A useful approach to estimating is for students to consider the size of unit fractions. To do this, students must first understand that the size of a fractional part decreases as the denominator increases. For example, to estimate the answer to $^9/_{10} + ^1/_8$, beginning students can be encouraged to reason that $^9/_{10}$ is almost 1, that $^1/_8$ is close to $^1/_{10}$, and that therefore the answer will be about 1. More advanced students can be encouraged to reason that $^9/_{10}$ is only $^1/_{10}$ away from 1, that $^1/_8$ is slightly larger than $^1/_{10}$, and therefore the solution will be slightly more than 1. The principle can and should be generalized beyond unit fractions once it is understood in that context. Key dimensions for generalization include estimating results of operations involving non-unit fractions (e.g., $^3/_4 \div ^2/_3$), improper fractions ($^7/_3 \div ^3/_4$), and decimals ($0.8 \div 0.33$).

Placement of Decimal Point. A common error when multiplying decimals, such as 0.8×0.9 or 2.3×8.7, is to misplace the decimal. Encouraging students to estimate the answer first can reduce such confusion. For example, realizing that 0.8 and 0.9 are both less than 1 but fairly close to it can help students realize that answers such as 0.072 and 7.2 must be incorrect.

Note. Taken from Example 3 on page 31 in the practice guide.

3. Address common misconceptions regarding computational procedures with fractions.

Instructional strategies from the examples

- Identify students who are operating with misconceptions about fractions, discuss the misconceptions with them, and make clear both why the misconceptions lead to incorrect answers and why correct procedures lead to correct answers.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1b, PS.2d
TEACHERS: INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.1, PLAN.SW.3, PLAN.Desc.1

Students' understanding of computational procedures with fractions can often be limited by misconceptions they carry with them about fractions. Once teachers have identified misconceptions in students, they can present these in discussions about why some procedures used by students result in correct answers, while others do not. Some common misconceptions are described here:

- **Believing that fractions' numerators and denominators can be treated as separate whole numbers.** A common mistake students make when adding or subtracting fractions is to try to add/subtract the numerators, then do the same with the denominators. This error is rooted in students applying their knowledge of whole numbers to fractions and not understanding that the denominator defines the size of the fractional part, while the numerator represents the number of parts of that size. Additionally, the fact that this approach works for the multiplication of fractions adds support for the misconception.

  To help students overcome this misconception, teachers should present meaningful problems to students, setting problems within real-world contexts that are relevant for their students. For example, page 32 in the practice guide presents the following: "If you have $^3/_4$ of an orange left and give $^1/_3$ of it to a friend, what fraction of the original orange do you have left?" In computing a solution, students should recognize that $\frac{3}{4} - \frac{1}{3} = \frac{2}{1}$ can't be correct because they can't come up with 2 oranges if they only started with $^3/_4$ of an orange in the first place. In using meaningful problems like this, teachers can help lay a foundation for students to think deeply about why treating the numerators and denominators as separate whole numbers is inappropriate, opening the door for discussion about appropriate procedures.

- **Failing to find a common denominator when adding or subtracting fractions with unlike denominators.** When adding or subtracting fractions with unlike denominators, a common mistake is that students just insert the larger denominator into the solution rather than convert both fractions to equivalent fractions with a common denominator. For example, when calculating $\frac{4}{5}+\frac{4}{10}$, students may find an incorrect solution of $\frac{8}{10}$. This misconception is rooted in students not understanding that different denominators represent different-sized unit fractions. A similar error of not converting the numerator of a fraction when changing the denominator is also rooted in this misconception (e.g., $\frac{2}{3}+\frac{2}{6}$ incorrectly becomes $\frac{2}{6}+\frac{2}{6}$). Teachers can use visual representations that show equivalent fractions (e.g., number lines, fraction strips) to help students see both the need for common denominators (i.e., equal sizes of unit-fraction parts) and appropriate changes to numerators.

- **Believing that only whole numbers need to be manipulated in computations with fractions greater than one.** Mixed numbers provide an additional challenge for students with misconceptions about fractions. Often, in addition/subtraction problems with mixed numbers, students may ignore the fractional part and work only with the whole numbers (e.g., $2\frac{2}{3}+5\frac{1}{2}=7$). Errors such as these may result from students ignoring the part of the problem they do not understand, misunderstanding the meaning of mixed numbers, or assuming that such problems simply have no solution.

  Related to this, students may think that the whole-number portion of a mixed number has the same denominator as the fraction in a problem. This misconception might lead students to incorrectly translate the problem $5-\frac{2}{3}$ into $\frac{5}{3}-\frac{2}{3}$. This misconception might also lead students to want to add the whole number to the numerator in more complex problems. Page 32 in the practice guide presents the following example of this type of error: $3\frac{1}{3}\times\frac{6}{7}=\left(\frac{3}{3}+\frac{1}{3}\right)\times\frac{6}{7}=\frac{4}{3}\times\frac{6}{7}=\frac{24}{21}$.

  To help overcome this misconception, teachers should help students understand the relationship between mixed numbers and improper fractions. Understanding how to translate each into the other is crucial for working with these fractions.

- **Treating the denominator the same in fraction addition and multiplication problems.** Misconceptions about fractions also lead to errors like students mixing up procedures for adding and multiplying fractions. This often results in computational errors, such as students leaving the denominator unchanged in fraction multiplication problems involving fractions with common denominators (e.g., $\frac{2}{5}\times\frac{3}{5}=\frac{6}{5}$). Errors such as this might also result from the fact that students see more fraction addition problems than multiplication problems, leading them to generalize procedures from fraction addition incorrectly to multiplication.

Teachers can address this misconception by helping students focus on the conceptual basis for fraction multiplication. For example, the product $\frac{1}{2} \times \frac{1}{2}$ can be thought of as "half of a half" and students can visualize that half of one half would equal one fourth (i.e., $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$). Additionally, teachers can help students think about the fact that multiplying by a fraction less than 1 results in a product that is smaller.

- **Failing to understand the invert-and-multiply procedure for solving fraction division problems.** When students lack a conceptual understanding, they often misapply procedures like "invert and multiply" (see Example 6 below). Errors like this often reflect a lack in conceptual understanding regarding why the invert-and-multiply procedure translates a multistep calculation into a more efficient way to calculate the correct quotient.

Example 6. Common student errors with the invert-and-multiply procedure

Problem
$\frac{2}{3} \div \frac{4}{5}$
Common Student Errors
Error 1—not inverting either fraction. Students may understand the procedure involves multiplying the two fractions but forget the "invert" part. $\frac{2}{3} \div \frac{4}{5} = \frac{8}{15}$ **Error 2—inverting the wrong fraction.** $\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{4}{5} = \frac{12}{10}$ **Error 3—inverting both fractions.** $\frac{2}{3} \div \frac{4}{5} = \frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$

Note. Taken from the examples on page 33 in the practice guide.

Teachers should ensure students understand the multistep calculation that is the basis of this procedure. The invert-and-multiply procedure is based in two mathematical concepts: (1) multiplying a number by its reciprocal results in a product of 1, and (2) dividing any number by 1 leaves the number unchanged. Teachers can show students how these connect to the invert-and-multiply procedure as follows for the problem $\frac{2}{3} \div \frac{4}{5}$:

- Multiplying both the dividend ($\frac{2}{3}$) and divisor ($\frac{4}{5}$) by the reciprocal of the divisor yields ($\frac{2}{3} \times \frac{5}{4}$) $\div$ ($\frac{4}{5} \times \frac{5}{4}$).
- Multiplying the original divisor ($\frac{4}{5}$) by its reciprocal ($\frac{5}{4}$) produces a divisor of 1, which results in $\frac{2}{3} \times \frac{5}{4} \div 1$, which yields $\frac{2}{3} \times \frac{5}{4}$.
- Thus, the invert-and-multiply procedure, multiplying $\frac{2}{3} \times \frac{5}{4}$, provides the solution.

4. Present real-world contexts with plausible numbers for problems that involve computing with fractions.

Instructional strategies from the examples

- Use real-world contexts that provide meaning to the fractions involved in the problem.
- Tailor problems around details that are familiar and meaningful to the students; gather ideas from them.
- Make connections between a real-world problem and the fraction notation used to represent it.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, 6.RP.3, 7.RP.3, 8.EEI.5
TEACHERS: INST.MS.1, INST.PIC.2, INST.AM.6, PLAN.SW.3

Teachers should present students with problems that use plausible numbers in real-world contexts. The contexts should also provide meaning to both the fraction quantities involved and the computational procedures used to find a solution. Setting problems in real-world measurement contexts (e.g., using rulers, ribbons, or measuring tapes) and using food in the problem can help. When including food items, teachers should use both discrete (e.g., boxes of candy, apples) and continuous items (e.g., pizza, candy bars). A good source of ideas for relevant contexts are the students themselves, as they will help tailor problems around details familiar and meaningful to them (e.g., school events, field trips, activities in other subjects).

When providing real-world problems and contexts, teachers should try to help students connect the problem with the fraction notation used to represent it. At times, students can correctly solve a problem presented in a real-world context but struggle to solve the same problem when presented in formal notation. Teachers should continue to connect the notation back to the real-world story problem as students work through the solution.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Students make computational errors (e.g., adding fractions without finding a common denominator) when using certain pictorial and concrete object representations to solve problems that involve computation with fractions.	Teachers should carefully choose representations that map easily and most directly to the fraction computation they are teaching (e.g., demonstrating the need for similar units when adding fractions), as use of some representations can actually reinforce misconceptions. To reinforce the need for common unit fractions, teachers should consider using representations that hold units constant (e.g., measuring tapes).
When encouraged to estimate a solution, students still focus on solving the problem via a computational algorithm rather than estimating it.	Teachers should present estimation as a preliminary tool for helping anticipate the relative size and appropriateness of a solution, not a shortcut to an answer. In this, teachers should help students focus on the reasoning needed to estimate a solution. Posing problems that are not quickly solvable with mental computation (e.g., $\frac{5}{9}+\frac{3}{7}$ instead of $\frac{5}{8}+\frac{3}{8}$) will help avoid this roadblock.

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15

This document provides a summary of Recommendation 4 from the WWC practice guide *Developing Effective Fractions Instruction for Kindergarten Through 8th Grade*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Minimal**

Recommendation

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

"Thinking proportionally" means that students understand the multiplicative relationship between two quantities and is a critical skill students must develop in preparation for success in advanced mathematics. Ratios, rates, and proportions are contexts that require understanding of multiplicative relationships and lead into the cross-multiplication algorithm. Teachers should develop students' proportional reasoning skills before teaching the cross-multiplication algorithm using a progression of problems that help build their informal reasoning strategies. Teachers should also make sure to return to the informal reasoning strategies after teaching the cross-multiplication algorithm, demonstrating that the algorithm and the informal reasoning students use lead to the same answers. As a caution, studies involving many types of problem-solving have demonstrated that students often learn a strategy to solve a problem in one context but fail to successfully generalize to other contexts.

How to carry out the recommendation

1. Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples

- Use a progression of problems that builds on students' developing strategies for proportional reasoning.
- Encourage students to apply their own strategies, discuss the various strategies' strengths and weaknesses, and help them understand why a problem's solution is correct.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, 6.RP.1, 6.RP.2, 7.RP.2a
TEACHERS: INST.PIC.2, INST.TCK.2, PLAN.SW.3

Teachers should provide opportunities for students to solve ratio, rate, and proportion problems, building students' strategies for proportional reasoning, before teaching the cross-multiplication algorithm. While encouraging students to apply their own strategies, teachers should also introduce ways to solve these problems if students struggle with generating their own strategies. In this, teachers should discuss the various strengths and weaknesses of strategies and help students understand why the resulting solution is correct.

A possible progression teachers might use to guide students from informal proportional reasoning strategies to the cross-multiplication algorithm might look like this (see Example 1):

- **Buildup Strategy.** Initially pose story problems that allow students to use a buildup strategy where they repeatedly add the numbers within one ratio to solve a problem. In these problems, teachers should ensure the numbers in the ratios are integrally related so that one can be generated by repeatedly adding the numbers in the other (e.g., 2:3 and 10:15). These should begin with problems involving smaller numbers to allow students to build their understanding, then progress to problems with larger numbers to demonstrate how time-consuming it can be to repeatedly add to these large numbers. This will help students recognize the value of using multiplication and division.

- **Unit Ratio Strategy.** Next, teachers can present problems that cannot be easily solved through repeated addition or through multiplying/dividing by a single integer (e.g., $\frac{x}{6} = \frac{3}{9}$). Solving these types of problems involves reducing the known ratio to a form with a numerator of 1, then determining the multiplicative relationship between the denominator in the new unit ratio and that in the ratio with the unknown value. This relationship can then be used to solve for the unknown value. This strategy can also be used in problems where the solution is not a whole number and bridge into helping students see the value in the cross-multiplication algorithm.
- **Cross-Multiplication.** Building on students' understanding of the unit ratio strategy, teachers can present problems that do not involve integral relations or that use ratios that cannot be easily reduced to unit fractions. This will help students see the advantages of using a strategy that can help solve problems, regardless of the numbers involved. Teachers should continue to help students make connections by having them return to previous strategies to see that cross-multiplication results in the same answer and discussing why this is the case.

Teachers should continue to present problems that can be solved easily through informal reasoning and mental mathematics, as well as those more easily solved with cross-multiplication. In this, teachers can discuss with students how to anticipate which approach might be easiest for a given problem.

Example 1. Problems encouraging specific strategies

Buildup Strategy
Sample problem. If Steve can purchase 3 baseball cards for $2, how many baseball cards can he purchase with $10? **Solution approach.** Students can build up to the unknown quantity by starting with 3 cards for $2, and repeatedly adding 3 more cards and $2, thus obtaining 6 cards for $4, 9 cards for $6, 12 cards for $8, and, finally, 15 cards for $10.

Unit Ratio Strategy

Sample problem. Yukari bought 6 balloons for $24. How much will it cost to buy 5 balloons?

Solution approach. Students might figure out that if 6 balloons costs $24, then 1 balloon costs $4. This strategy can later be generalized to one in which eliminating all common factors from the numerator and denominator of the known fraction does not result in a unit fraction (e.g., a problem such as $\frac{6}{15} = \frac{x}{10}$, in which reducing $\frac{6}{15}$ results in $\frac{2}{5}$).

Cross-Multiplication

Sample problem. Luis usually walks the 1.5 miles to his school in 25 minutes. However, one of the streets on his usual path is being repaired today, so he needs to take a 1.7-mile route. If he walks at his usual speed, how much time will it take him to get to his school?

Solution approach. This problem can be solved in two stages. First, because Luis is walking at his "usual speed," students know that $\frac{1.5}{25} = \frac{1.7}{x}$. Then, the equation may be most easily solved using cross-multiplication. Multiplying 25 and 1.7 and dividing the product by 1.5 yields the answer of $28\frac{1}{3}$ minutes, or 28 minutes and 20 seconds. It would take Luis 28 minutes and 20 seconds to reach school using the route he took today.

Note. Taken from Example 4 on page 38 of the practice guide.

Example 2. Why cross-multiplication works

Teachers can explain why the cross-multiplication procedure works by starting with two equal fractions, such as $\frac{4}{6} = \frac{6}{9}$. The goal is to show that when two equal fractions are converted into fractions with the same denominator, their numerators also are equivalent. The following steps help demonstrate why the procedure works.

Step 1. Start with two equal fractions, for example: $\frac{4}{6} = \frac{6}{9}$.

Step 2. Find a common denominator using each of the two denominators.

a. First, multiply $\frac{4}{6}$ by $\frac{9}{9}$, which is the same as multiplying $\frac{4}{6}$ by 1.

b. Next, multiply $\frac{6}{9}$ by $\frac{6}{6}$, which is the same as multiplying $\frac{6}{9}$ by 1.

Step 3. Calculate the result: $\frac{(4\times9)}{(6\times9)} = \frac{(6\times6)}{(9\times6)}$

Step 4. Check that the denominators are equal. If two equal fractions have the same denominator, then the numerators of the two equal fractions must be equal as well, so $4 \times 9 = 6 \times 6$.

Note that in this problem, $4 \times 9 = 6 \times 6$ is an instance of $(a \times d = b \times c)$.

As a result, students can see that the original proportion, $\frac{4}{6} = \frac{6}{9}$, can be solved using cross-multiplication, $4 \times 9 = 6 \times 6$, as a procedure to create equivalent ratios efficiently.

Note. Taken from Example 5 on page 39 of the practice guide.

2. Encourage students to use visual representations to solve ratio, rate, and proportion problems.

Instructional strategies from the examples

- Select representations that are likely to elicit insight into a particular aspect of ratio, rate, and proportion concepts.
- Encourage students to create their own representations.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.1c, PS.2b, PS.4a, PS.5a, 6.RP.2a, 6.RP.3b, 7.RP.2, 8.EEI.5c
TEACHERS: INST.PIC.2, INST.TCK.2

Teachers should encourage students to use visual representations when solving ratio, rate, and proportion problems and select representations that highlight specific concepts within the problem. For example, teachers can use a ratio table to represent the relations in a proportion problem and provide specific reference to highlight how multiplication leads to the same solution as the buildup strategy (see Example 3). In addition, teachers can use ratio tables to help students explore different aspects of proportional relationship, such as multiplicative relationships within and between ratios (see Example 3).

Teachers should encourage students to develop their own representations and not always provide them. With ratios, rates, and proportions, students initially tend to use tabular or other systematic record-keeping formats. Through formal instruction and exposure, teachers can introduce other representations and encourage students to use these in future problems.

Example 3. Ratio table for a proportion problem

Problem
How many cups of flour are needed for 32 people when a recipe calls for 1 cup of flour to serve 8 people?

Solution

Students can use a ratio table to repeatedly add 1 cup of flour per 8 people to find the correct amount for 32 people.

Cups of Flour	1	2	3	4
Number of People Served	8	16	24	32

Students can also use the ratio table to see that multiplying by the ratio $\frac{4}{4}$ (i.e., four times the recipe) provides the amount of flour needed for 32 people. Alternatively, the number of people served is always 8 times the number of cups of flour needed; thus, the ratio between them is 1:8.

Note. Adapted from Example 9 on page 39 and Example 10 on page 40 of the practice guide.

3. Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples

- Focus instruction on the meaningful features of different problem types so students can transfer their learning to new situations.
- Help students identify key information needed to solve a problem and how to use diagrams or pictures to depict that information.
- Encourage students to use different diagrams and strategies to arrive at solutions.
- Provide opportunities for students to compare and discuss their diagrams and strategies.
- Provide real-life contexts in problems.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.1d, PS.2c, PS.2d, PS.3b, PS.3d, PS.5a, PS.7b, 7.RP.2, 7.RP.2d
TEACHERS: INST.MS.2, INST.AM.4, INST.AM.9, INST.TCK.2, PLAN.SW.1, PLAN.Desc.1

Teachers should focus instruction on meaningful features of different ratio, rate, and proportion problems to help students identify problems with common underlying structures. The goal is for students to transfer their learnings to new situations and contexts. For example, a recipe problem might call for 3 eggs to make 20 cupcakes and ask students to find the number of eggs for 80 cupcakes, while in another problem, building 3 doghouses requires 42 boards, and students need to determine how many boards are needed for 9 doghouses.

To develop students' ability to generalize to other contexts, teachers should first help them identify key information they will need to solve the problem. Then, teachers can teach students how to use diagrams to represent that information, focusing the diagrams on not only depicting the information but also the relationships between different quantities in the problem. Teachers should also provide opportunities for students to compare and discuss their various diagrams and strategies.

Finally, teachers should set ratio, rate, and proportion problems within real-life contexts, such as unit price, scaling, recipes, mixture, and time/speed/distance.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Many students misapply the cross-multiplication strategy.	Teachers can carefully present several examples of why cross-multiplication works, following the process in Example 2, to help students understand the logic behind the procedure. This will also help students see why the correct form of a ratio problem is necessary for the procedure to work.
Some students rely nearly exclusively on the cross-multiplication strategy for solving ratio, rate, and proportion problems, failing to recognize that there often are more efficient ways to solve these problems.	Teachers should encourage a variety of strategies for solving ratio, rate, and proportion problems. Presenting problems that are easier to solve with strategies other than cross-multiplication encourages students to use prior knowledge. For example, to find the solution for $\frac{5}{15} = \frac{6}{x}$, students can see that 15 is a multiple of 5 (i.e., $5 \times 3 = 15$), so *x* will be the same multiple of 6 (i.e., $x = 6 \times 3$). Requiring students to solve problems mentally can also help them see and use other strategies, as well as build number sense.
Students do not generalize strategies across different ratio, rate, and proportion contexts.	When presenting problems set in different contexts, teachers should also make sure to link these new problems with problems students have previously solved. Teachers can also encourage students to judge whether the same solution strategy can be used for different types of problems (e.g., recipe and mixture problems can be organized the same way so solutions can be compared side by side).

Reference: Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15

This document provides a summary of Recommendation 4 from the WWC practice guide *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Strong**

Recommendation

Interventions should include instruction on solving word problems that is based on common underlying structures.

Students who struggle with mathematics often face even larger difficulties in solving word problems. Thus, teachers should provide systematic, explicit instruction on solving word problems that is based in the problems' underlying structure. When students are taught the underlying structure of a word problem, they not only have greater success in problem-solving but can also gain insight into the deeper mathematical ideas in word problems. In this, teachers should make explicit connections between the structures of familiar and unfamiliar problems to help students learn when to apply a previously learned solution strategy.

How to carry out the recommendation

1. Teach students about the structure of various problem types, how to categorize problems based on structure, and how to determine appropriate solutions for each problem type.

Instructional strategies from the examples

- Explicitly teach the important features of groups of problems with similar mathematical structures.
- Use visualizations to help students identify similarities in the structure of problems to determine appropriate solutions.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.7c
TEACHER: INST.PIC.2, INST.AM.4, INST.TCK.2

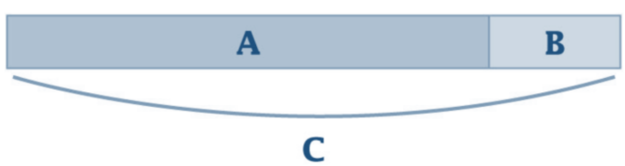
For groups of problems with similar mathematical structures, referred to as problem types, teachers should explicitly teach students the important underlying structural features that relate problems in the same problem type. Change problems (see Example 1) and compare problems (see Example 2) are two examples. Change problems always include a time element, and students use addition or subtraction to determine how much more or less. Compare problems focus on making comparisons between two different sets, and students need to calculate the unknown difference, unknown compared amount, or unknown referent amount.

To build understanding of each problem type, teachers should initially teach solution rules (i.e., guiding questions that lead to a solution equation) for each problem type through fully and partially worked examples, followed by student practice in pairs. Additionally, using visual representations can help students see the problem structure and determine an appropriate solution method.

Example 1. Sample problem types

Change Problems

The two problems here are addition and subtraction problems that students may be tempted to solve using an incorrect operation. In each case, students can draw a simple diagram like the one shown below, record the known quantities (two of three of A, B, and C), and then use the diagram to decide whether addition or subtraction is the correct operation to use to determine the unknown quantity.



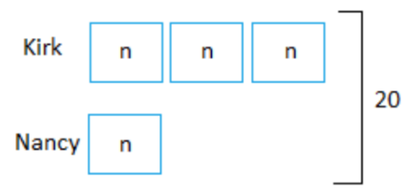
Problem 1. Brad has a bottlecap collection. After Madhavi gave Brad 28 more bottlecaps, Brad had 111 bottlecaps. How many bottlecaps did Brad have before Madhavi gave him more?

Problem 2. Brad has a bottlecap collection. After Brad gave 28 of his bottlecaps to Madhavi, he had 83 bottlecaps left. How many bottlecaps did Brad have before he gave Madhavi some?

Compare Problems

Problem. Kirk has 3 times as many baseball cards as Nancy. Together, they have 20 baseball cards. How many cards does Kirk have?

Visual representation.



Note. Adapted from Example 1 on page 27 and Example 2 on page 28 of the practice guide.

2. Teach students to recognize the common underlying structure between familiar and unfamiliar problems and to transfer known solution methods from familiar to unfamiliar problems.

Instructional strategies from the examples

- Explicitly show students that not all pieces of information in the problem may be important for identifying the underlying structure.
- Provide opportunities for students to explain and discuss why a piece of information is relevant or irrelevant.

South Carolina standards alignment

MATHEMATICS: PS.1b, PS.1c, PS.4b, PS.7c
TEACHERS: INST.MS.2, INST.PIC.2, INST.AM.4, INST.TCK.2, INST.TH.2, PLAN.SW.1, PLAN.SW.3

Superficial changes (e.g., format, key vocabulary, the inclusion of irrelevant information) can often lead students to see a familiar problem as one that is new and unfamiliar. Format changes might be something like presenting a problem as an advertisement in a brochure rather than in a traditional paragraph form. Changes in key vocabulary might be identifying a fraction as half, one half, or $\frac{1}{2}$. These superficial changes are irrelevant to solving the problem but often lead to students struggling with identifying common underlying structures between new and old problems.

To facilitate transfer of methods to new problems from old problems, teachers should explicitly show students that not all pieces of information are relevant to identifying the underlying structure. Teachers should also provide opportunities for students to explain and discuss why a certain piece of information is relevant or irrelevant.

Example 2. Different problems with the same strategy

Problem 1. Mike wants to buy 1 pencil for each of his friends. Each packet of pencils contains 12 pencils. How many packets does Mike have to buy to give 1 pencil to each of his 13 friends?

Problem 2. Mike wants to buy 1 pencil for each of his friends. Sally wants to buy 10 pencils. Each box of pencils contains 12 pencils. How many boxes does Mike have to buy to give 1 pencil to each of his 13 friends?

Discussion. The structure of these two problems is identical, as is the required solution. However, the inclusion of irrelevant information in the second problem (i.e., Sally wants to buy 10 pencils) may cause students to see these two problems as different. Teachers should provide an opportunity for students to discuss what information in the problem is relevant to finding a solution or, if the students struggle to identify what is relevant and irrelevant, explicitly show students the relevant information.

Note. Adapted from Example 3 on page 39 of the practice guide.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
The curricular material may not classify problems into problem types.	The key issue is determining the problem types and an instructional sequence for teaching them so that students understand a set of problem structures and the related mathematics. To accomplish this, teachers may need assistance from a mathematics coach, a mathematics specialist, or a district or state curriculum guide.
As problems get complex, so will the problem types and the task of discriminating among them.	As problems become more complex (e.g., multistep problems), teachers may need to explicitly and systematically teach students how to differentiate one problem type from another. Again, teachers themselves may need additional support in determining problem types, justifying their responses, and explaining and modeling problem types to students.

Reference: Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools (NCEE 2009-4060). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/2

This document provides a summary of Recommendation 5 from the WWC practice guide *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools*. Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **K–8**

LEVEL OF EVIDENCE: **Moderate**

Recommendation

Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.

Many students struggle with mathematics as a result of not having a strong understanding of the relationships between the abstract symbols used in mathematics and various visual representations. Helping students develop their ability to express mathematical ideas using visual representations, then correctly convert this into symbols is critical. As such, teachers should systematically teach students how to both develop visual representations and translate these into standard mathematical symbols they will use in solving the problem.

How to carry out the recommendation

1. Use visual representations such as number lines, arrays, and strip diagrams.

Instructional strategies from the examples

- Use visual representations (e.g., number lines, diagrams, pictorial representations) extensively and consistently.
- Explicitly link visual representations with standard mathematical symbols.

South Carolina standards alignment

MATHEMATICS: PS.2a, PS.2b, PS.4a
TEACHERS: INST.PIC.2, INST.TCK.2

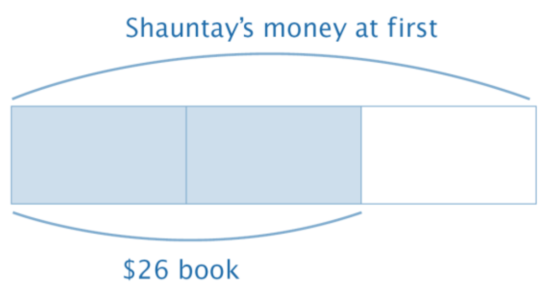
In early grades, teachers can use number lines, number paths, and other pictorial representations to help students develop foundational skills and procedural operations for counting, addition, and subtraction. In upper grades, diagrams and pictorial representations can be used to teach fractions or help students make sense of the underlying mathematical structure in word problems.

Example 1. Using strip diagrams to make sense of fractions

Problems
Shauntay spent $^2/_3$ of the money she had on a book that cost \$26. How much money did Shauntay have before she bought the book?

Solution

Strip diagrams (also called model diagrams and bar diagrams) are one type of diagram that can be used to represent and help solve this problem. The full rectangle (consisting of all three equal parts joined together) represents Shauntay's money before she bought the book. Since she spent $^2/_3$ of her money on the book, two of the three equal parts represent the $26 she spent on the book.



Two parts of the strip diagram represent the $26 spent on the book. That means that one part would equal $13 ($\$26 \div 2 = \$13$). To find the amount she had at first, we would multiply $13 by 3. So Shauntay had $39 before she bought the book.

Note. Adapted from Example 6 on page 34 of the practice guide.

2. If visuals are not sufficient for developing accurate abstract thought and answers, use concrete manipulatives first. Although this can also be done with students in upper elementary and middle school grades, use of manipulatives with older students should be expeditious because the goal is to move toward understanding of—and facility with—visual representations, and finally, to the abstract.

Instructional strategies from the examples

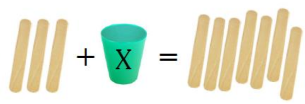
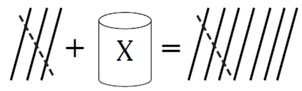
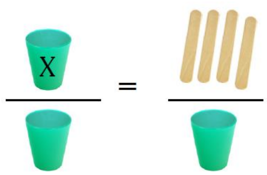
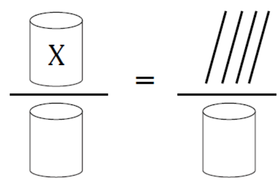
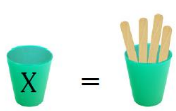
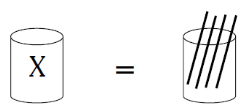
- Use manipulatives, when necessary, and focus on systematically phasing them out.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a, PS.2b, PS.4a
TEACHERS: INST.PIC.2, INST.AM.4, INST.TCK.2, PLAN.SW.3

Teachers should use concrete objects when using visual representations doesn't sufficiently support students understanding the more abstract symbols in math. However, teachers should focus on systematically fading the use of manipulatives to guide students toward reaching the more abstract level of using mathematical symbols. This involves explicitly teaching the concepts and operations with the concrete manipulatives and consistently connecting the visual with the abstract levels. Teachers should use consistent language across the different representations (concrete manipulatives, visual representation, and abstract symbols) to solidify connections for students.

Example 2. A set of matched concrete, visual, and abstract representations to teach solving single-variable equations

Solving the Equation with Concrete Manipulatives (Cups and Sticks)	Solving the Equation with Visual Representations of Cups and Sticks	Solving the Equation with Abstract Symbols
A 		$3 + 1x = 7$
B 		$-3 \qquad -3$
C D 		$\dfrac{1x}{1} = \dfrac{4}{1}$
E 		$x = 4$

Note. Taken from Example 8 on page 35 of the practice guide.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Many intervention materials provide very few examples of the use of visual representations.	Because many curricular materials do not include sufficient examples of visual representations, the interventionist may need the help of the mathematics coach or other teachers in developing the visuals. District staff can also arrange for the development of these materials for use throughout the district.
Some teachers or interventionists believe that instruction in concrete manipulatives requires too much time.	Expeditious use of manipulatives cannot be overemphasized. Since tiered interventions often rely on foundational concepts and procedures, the use of instruction at the concrete level allows for reinforcing and making explicit the foundational concepts and operations. Note that overemphasis on manipulatives can be counterproductive because students manipulating only concrete objects may not be learning to do math at an abstract level. The interventionist should use manipulatives in the initial stages strategically and then scaffold instruction to the abstract level. So, although it takes time to use manipulatives, this is not a major concern because concrete instruction will happen only rarely and expeditiously.
Some interventionists may not fully understand the mathematical ideas that underlie some of the representations. This is likely to be particularly true for topics involving negative numbers, proportional reasoning, and interpretations of fractions.	It is perfectly reasonable for districts to work with a local university faculty member, high school mathematics instructor, or mathematics specialist to provide relevant mathematics instruction to interventionists so they feel comfortable with the concepts. This can be coupled with professional development that addresses ways to explain these concepts in terms their students will understand.

Reference: Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools* (NCEE 2009-4060). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/2

This document provides a summary of Recommendation 6 from the WWC practice guide Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools. Full reference at the bottom of last page.

CONTENT: Mathematics

GRADE LEVEL(S): K–8

LEVEL OF EVIDENCE: Moderate

Recommendation

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

Students with difficulties in mathematics often demonstrate lack of fluency with quick retrieval of basic arithmetic fact (e.g., $3 \times 9 =$ __ and $11 - 7 =$ __). This weak ability is likely to inhibit students' understanding of deeper mathematics concepts (e.g., rational numbers, the commutative property). For struggling students, devoting 10 minutes each day to building proficiency with quick retrieval of arithmetic facts can enhance their ability to grasp more advanced mathematics concepts.

How to carry out the recommendation

1. *Provide about 10 minutes per session of instruction to build quick retrieval of basic arithmetic facts. Consider using technology, flash cards, and other materials for extensive practice to facilitate automatic retrieval.*

Instructional strategies from the examples

- *Present facts in number families.*
- *Integrate previously learned facts into practice activities.*
- *Provide enough practice so retrieval becomes automatic.*

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

South Carolina standards alignment

- ***MATHEMATICS:*** *None*
  TEACHERS: *INST.AM.10, INST.TCK.2*

  The goal for students is the quick retrieval of facts without the use of pencil and paper or manipulatives. Teachers should present facts in number families (e.g., $7 \times 8 = 56$, $8 \times 7 = 56$, $56 \div 7 = 8$, and $56 \div 8 = 7$) to help build student's fluency. This also helps students learn about inverse operations. Additionally, teachers should incorporate cumulative review into these activities, integrating previously learned facts into students' practice activities and providing enough practice so retrieval for students become automatic.

  Note. The panel that developed the practice guide acknowledges that students who are proficient in grade-level mathematics may not need to practice each session but might still benefit from periodic, cumulative review.

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

2. *Teach students in grades 2 through 8 how to use their knowledge of properties, such as commutative, associative, and distributive law, to derive facts in their heads. (Note: This is really step 3 in the practice guide, as step 2 focuses on K–2.)*

3. *Instructional strategies from the examples*

- *Guide students to use properties of arithmetic (e.g., composition, decomposition, distributive property) to solve complex facts involving multiplication and division.*

4. *South Carolina standards alignment*

- ***MATHEMATICS:*** *PS.1a, PS.2a*
  TEACHERS: *INST.AM.4, INST.TCK.2, PLAN.SW.3*

Rather than solely rely on rote memorization of facts, teachers should guide students to use what they know about properties of mathematics to master more complex facts about multiplication and division. Teachers can teach students how to use composition and decomposition, as well as the distributive property, to help increase students' facility with retrieving multiplication facts more quickly.

5. *Example of using mathematical properties to support multiplication facts*

- *To understand and quickly produce the seemingly difficult multiplication fact* $13 \times 7 =$*, students recall that* $13 = 10 + 3$*, something they should have been taught consistently during their elementary career. Then, since* $13 \times 7 = (10 + 3) \times 7 = 10 \times 7 + 3 \times 7$*, the fact is parsed into easier, known problems* $10 \times 7 =$ *and* $3 \times 7 =$ *by applying the distributive property. Students can then rely on the two simpler multiplication facts (which they had already acquired) to quickly produce an answer mentally.*

Note. Adapted from example in text on page 39 in the practice guide.

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
Students may find fluency practice tedious and boring.	Games that provide students with the opportunity to practice new facts and review previously learned facts by encouraging them to beat their previous high score can help the practice become less tedious. Players may be motivated when their scores rise and the challenge increases.
Curricula may not include enough fact practice or may not have materials that lend themselves to teaching strategies.	Some contemporary curricula deemphasize fact practice, so this is a real concern. In this case, teachers should consider using a supplemental program, based in either flash cards or technology.

Reference: Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools (NCEE 2009-4060). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/2

This document provides a summary of Recommendation 1 from the WWC practice guide *Teaching Math to Young Children.* Full reference at the bottom of last page.

CONTENT: **Mathematics**
GRADE LEVEL(S): **preK–K**
LEVEL OF EVIDENCE: **Moderate**

Recommendation

Teach numbers and operations using a developmental progression.

Understanding what skills and knowledge children already possess is the starting point for instruction. A developmental progression can provide a road map of the next steps. For example, when teaching numbers and operations, teachers should ensure that children proceed through each level of the developmental progression. At each level, children should begin by practicing with small sets of objects and progress to larger sets until they master the skills and knowledge at that level.

Example of a piece of developmental progression for number knowledge

Subitizing (small-number recognition)	Subitizing refers to a child's ability to immediately recognize the total number of items in a collection and label it with an appropriate number word. When children are presented with many different examples of a quantity (e.g., two eyes, two hands, two socks, two shoes, two cars) labeled with the same number word, as well as non-examples labeled with other number words (e.g., three cars), children construct precise concepts of one, two, and three.
	A child is ready for the next step when, for example, they can see one, two, or three stickers and immediately—without counting—state the correct number of stickers.
Meaningful object counting	Meaningful object counting is counting in a one-to-one fashion and recognizing that the last word used while counting is the same as the total (this is called the cardinality principle).
	A child is ready for the next step when, for example, if given five blocks and asked, "How many?" they count by pointing and assigning one number to each block: "One, two, three, four, five," and recognizes that the total is "five."

Counting-based comparisons of collections larger than three	Once children can use small-number recognition to compare small collections, they can use meaningful object counting to determine the larger of two collections (e.g., “seven” items is more than “six” items because you have to count further).
	A child is ready for the next step when they are shown two different collections (e.g., nine bears and six bears) and can count to determine which is the larger one (e.g., “nine” bears is more).
Number-after knowledge	Familiarity with the counting sequence enables a child to have number-after knowledge—i.e., to enter the sequence at any point and specify the next number instead of always counting from one.
	A child is ready for the next step when they can answer questions such as, “What comes after five?” by stating “five, six” or simply “six” instead of, say, counting “one, two, . . . six.”
Mental comparisons of close or neighboring numbers	Once children recognize that counting can be used to compare collections and have number-after knowledge, they can efficiently and mentally determine the larger of two adjacent or close numbers (e.g., that “nine” is larger than “eight”).
	A child has this knowledge when they can answer questions such as, “Which is more, seven or eight?” and can make comparisons of other close numbers.
Number-after equals one more	Once children can mentally compare numbers and see that “two” is one more than “one” and that “three” is one more than “two,” they can conclude that any number in the counting sequence is exactly one more than the previous numbers.
	A child is ready for the next step when they recognize, for example, that “eight” is one more than “seven.”

Note. Taken from Table 3 on page 13 of the practice guide.

How to carry out the recommendation

1. First, provide opportunities for children to practice recognizing the total number of objects in small collections (one to three items) and labeling them with a number word without needing to count them.

Instructional strategies from the examples

- Build children's understanding of quantity through subitizing activities.
- Have children identify sets that contain an equal number of objects without counting.
- Build children's ability to compare sets by having them find sets of objects that contain the same number of objects and those that do not (e.g., "three" and "not three").

South Carolina standards alignment

MATHEMATICS: K.NS.6
TEACHERS: No direct alignment

Children should be able to determine the number of objects in a small set without counting. This is known as *subitizing*. During classroom transitions, teachers may find small sets of objects (five or less) in the classroom and ask, "How many (object name) do you see?" After children can successfully identify three objects, they should be able to understand related sets that have the same number of objects. For example, three pencils and three erasers are two sets with the same number of objects. Once children have experience in recognizing sets of objects containing similar items, teachers can progress to sets with dissimilar items (for example, a set of three containing a pencil, a crayon, and an eraser). When developing subitizing, children may overgeneralize the term "three" or "four" to mean "many," so teachers should identify a set of three objects as "three" and a set of four objects as "not three" to help children recognize the difference. Teachers can then challenge children to find sets of objects around the classroom that are "three" or "not three." See Example 1 on page 16 of the practice guide referenced on the last page of this document.

2. Next, promote accurate one-to-one counting as a means of identifying the total number of items in a collection.

Instructional strategies from the examples

- Build on children's ability to subitize to help them develop one-to-one object counting, beginning with organized sets, then moving to unorganized sets.
- After successfully counting a set of objects, help children develop cardinality—recognizing that the last counting word represents the number of objects in the set.
- Capitalize on children's counting errors to strengthen their counting ability.

South Carolina standards alignment

MATHEMATICS: K.NS.5
TEACHERS: PLAN.SW.3

One-to-one counting occurs when children count with number words in a consecutive order to determine the number of objects in a set, using only one number name for each unique item. For example, a child counting a set of pens points at a pen and says, "One," then points at the next pen and says, "Two," and finally points at the last pen and says, "Three." Children should begin with small sets (one to four objects). Teachers should help them in realizing that the last number they count is the total number in a set. Then, children can progress to larger sets (four to 10 objects). Teachers should also demonstrate that order does not affect the result by using objects around the classroom and counting them in different ways. For example, when counting pens, no matter which pen children start with, they will reach the same result.

Example counting activity: The *Hidden Stars* game

Objective
Practice using one-to-one counting and the final number counted to identify "how many" objects. **Materials needed:** • Star stickers in varying quantities from one to 10, glued to 5-by-8-inch cards • Paper for covering cards **Directions:** Teachers can tailor the *Hidden Stars* game for use with the whole class, a small group, or individual children. Show children a collection of stars on an index

card. Have one child count the stars. Once the child has counted the stars correctly, cover the stars and ask, "How many stars am I hiding?"
Early math content areas covered
• Counting • Cardinality (using the last number counted to identify the total in the set)
Monitoring children's progress and tailoring the activity appropriately
• Work with children in a small group, noting each child's ability to count the stars with accuracy and say the amount using the cardinality principle (the last number counted represents the total). • When children repeat the full count sequence, model the cardinality principle. For example, for four items, if a child repeats the count sequence, say, "One, two, three, four. So I need to remember four. There are four stars hiding." • Have a child hide the stars while telling him or her how many there are, emphasizing the last number as the significant number.
Using the activity to increase math talk in the classroom
• Ask, "How many?" (e.g., "How many blocks did you use to build your house? How many children completed the puzzle?")

Note. Taken from Example 2 on page 18 of the practice guide.

Errors are to be expected when children learn how to count. Common counting errors include sequence errors, coordination errors, and keeping-track errors, as well as skimming and not recognizing cardinality. In sequence errors, children confuse the sequence of numbers. In this case, teachers might ask the children to sing the number sequence. If children are skipping specific numbers, teachers should focus on practicing that part of the sequence. Coordination errors involve children labeling an object with more than one word or pointing to an object without counting it. Teachers might correct these errors by encouraging children to slow down and count each object once. When children make keeping-track errors, they may count an object twice. Teachers can help children differentiate counted objects from uncounted objects by having them pick up the counted objects and place them to the side. For more information on common counting errors and recommendations on how to remedy them, see Table 4 on page 19 of the practice guide referenced on the last page of this document.

3. Once children can recognize or count collections, provide opportunities for children to use number words and counting to compare quantities.

Instructional strategies from the examples

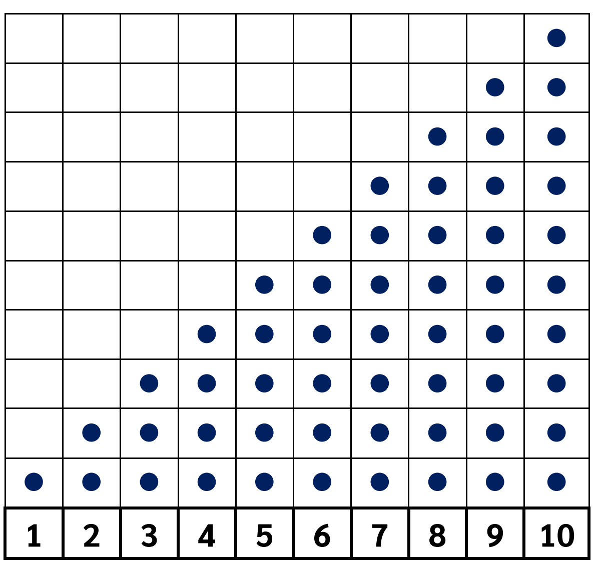
- Initially present children with sets in which one is obviously larger, then move to sets that are more equal in size that require counting or subitizing.

South Carolina standards alignment

MATHEMATICS: K.NS.7
TEACHERS: PLAN.IP.3, PLAN.SW.3

Children can progress in making meaningful comparisons of sets by, for example, identifying "more" and "fewer." First, teachers should present children with two sets of objects, one of which is obviously larger, and ask them which set has more or fewer objects. Next, teachers can demonstrate that the further in the counting sequence children count, the larger the numbers become. The sample chart below provides a visual of this increase in size or quantity and can be helpful. Real-world examples, such as counting to determine who has more points in a game, can also help children increase their ability to compare quantities. Also helpful is knowing what number comes next without counting (number-after relations). Teachers can reinforce this knowledge by asking children such questions as "This is the sixth pen, so the next pen will be how many?"

Sample cardinality chart



Note. Taken from page 20 of the practice guide.

4. Encourage children to label collections with number words and numerals.

Instructional strategies from the examples

- Label sets of objects with representations of quantity, numeral, and number words to help children connect the different representations.

South Carolina standards alignment

MATHEMATICS: K.NS.4a
TEACHERS: PLAN.SW.3

Numerals are a way to represent quantity. To help children understand this, teachers should pair number words with numerals. For example, "3," "three," and "three objects" have the same meaning. Teachers can label sets of objects that children see in the classroom. For example, a set of three pens can be labeled "3, • • •, three." The three dots provide scaffolding and represent the quantity for children who do not recognize numerals yet.

Example activity: The *Concentration: Numerals and Dots* game

Objective
Match numerals with corresponding quantities. **Materials needed:** • One set of 20 cards: 10 cards with numerals from 1 to 10 along with the corresponding number of dots, and 10 cards with pictures of objects (the numbers of objects corresponding to a numeral from 1 through 10). • For even more advanced play, once children are proficient at numerals 1–10, teachers can create cards for numerals 11–20. **Directions:** Half of the cards have a numeral and dots to represent the amount (e.g., the numeral 3 and three dots) on one side, and the other half have pictures of collections of objects on one side (e.g., three horses, four ducks). The other side of each card is blank. The cards are placed facedown, with the numeral cards in one area and the picture cards in another. A player chooses one numeral card and one picture card. If they match, then the player keeps those cards. Play continues until no further matching cards remain. The player with the most cards wins the game.
Early math content areas covered
• Numeral recognition.

• Corresponding quantity. • If the objects in the pictures on the cards are in different orders, it can help reinforce the idea that appearance does not matter when it comes to numbers.
Monitoring children's progress and tailoring the activity appropriately
• Play the game with a small group of children, noting each child's progress in practicing and achieving the objectives. • This game can be played with children who are not familiar with numeracy concepts. • Use fewer cards, lower numbers, or cards with dots to scaffold. As children gain proficiency with the concepts, increase the number of cards and the size of the numbers.
Using the activity to increase math talk in the classroom
• Before asking, "How many?" ask, "How can we find out how many?"

Note. Taken from Example 3 on page 22 of the practice guide.

5. Once children develop these fundamental number skills, encourage them to solve basic problems.

Instructional strategies from the examples

- Provide opportunities for children to explore the effects of adding or removing one object from a set using counting strategies.
- Once children have developed some facility, present situations where the final result is hidden.

South Carolina standards alignment

MATHEMATICS: K.PS.1, K.PS.4, K.ATO
TEACHERS: PLAN.IP.3, PLAN.SW.3

Children should proceed to develop an understanding of the effects of altering the number of objects in a set as they prepare for math problem-solving. To start, children can remove or add one object, recount, and review how the number of objects has changed. They can also use counting strategies in problem-solving activities in the classroom, such as counting the number of groups to determine how many whiteboards to hand out.

After children have had opportunities to explore adding and subtracting objects from sets, teachers can move to situations where the final results are hidden from view. For example, teachers can show children a set of four pens, cover the pens with a cloth, take one pen from underneath the cloth, and then ask the children to determine how many pens are left under the cloth. Once the children decide how many objects are left, teachers can remove the cloth and have the children count to see if they solved the problem correctly. Snack time is also a great opportunity for children to apply counting skills. Teachers can ask, “How many will you have after you eat one snack item?” or, “How many will you have after your friend gives you one snack item?” to have children problem-solve without seeing the end set.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
I want to provide strong math foundations for my children, but I am not really comfortable with math myself.	Teachers who feel less comfortable with math should base classroom projects on real-world examples. Setting up a toy store in the classroom provides a more comfortable setting in which to integrate lessons. Any activity that is of interest and involves counting presents an opportunity to build children's math skills.
Each child in the class is at a different level in the developmental progression I am using to guide instruction.	Teachers might split children into groups, using the developmental progression to create groups of children at a similar level. Dividing children into groups allows teachers to assign tasks based on each group's level of proficiency. Teachers can also create groups containing children at diverse levels of proficiency. This allows for children at a higher level to model a skill for others.
A child is stuck at a particular point in the developmental progression.	If children are stuck, they have likely not yet mastered a skill from an earlier level in the developmental progression. Teachers should use the developmental progression to help identify the unmastered skill and provide opportunities for children to practice it further before returning to the point at which they were stuck.

Reference: Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children* (NCEE 2014-4005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/18

This document provides a summary of Recommendation 2 from the WWC practice guide *Teaching Math to Young Children.* Full reference at the bottom of last page.

CONTENT: **Mathematics**
GRADE LEVEL(S): **preK–K**
LEVEL OF EVIDENCE: **Minimal**

Recommendation

Teach geometry, patterns, measurement, and data analysis using a developmental progression.

Understanding what skills and knowledge children already possess is the starting place for instruction. A developmental progression can provide a road map of next steps. To ensure that children have early opportunities to experience a wide range of math content, teachers should use a developmental progression to expose them to geometry and data. Teachers should ensure that children proceed through each level of the developmental progression. Helping children build understanding beyond numbers and operations increases their likelihood of success in later math.

How to carry out the recommendation

1. Help children recognize, name, and compare shapes, and then teach them to combine and separate shapes.

Instructional strategies from the examples

- Have children find and name shapes in the world around them.
- Identify critical attributes of shapes, focusing on precise mathematical language.
- Provide both examples and non-examples of shapes.
- Present children with opportunities to combine and/or separate shapes to construct other shapes.

South Carolina standards alignment

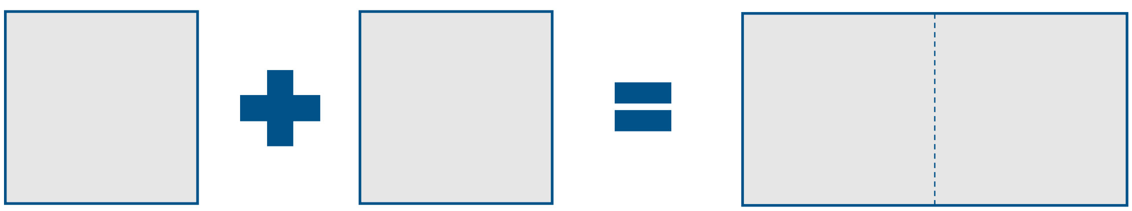
MATHEMATICS: K.G.2, K.G.4, 1.G.2, 1.G.3, 1.G.4
TEACHERS: No direct alignment

Teachers should begin by helping children find and name shapes in their own environments. Once children have developed confidence in naming shapes, they should be challenged to name the most important characteristics or “critical attributes” of each shape using standard vocabulary. For example, a critical attribute of a triangle is that it has three sides. Teachers should also point out characteristics that are not critical attributes. For instance, teachers should note that all sides of a triangle do not need to have equal length.

To solidify children’s understanding, teachers should provide both examples and non-examples of shapes. A non-example is a shape that lacks one or more of the critical characteristics. For a shape activity that highlights critical attributes, see Example 4 on page 29 in the practice guide referenced on the last page of this document.

Once children understand the fundamentals of shapes, teachers should ask them to explore how shapes can be combined and separated. For example, combining two identical squares can create a rectangle, or cutting a triangle across the middle may make two triangles.

Combining two identical squares will make a rectangle



Note. Taken from Figure 4 on page 28 of the practice guide.

2. Encourage children to look for and identify patterns, and then teach them to extend, correct, and create patterns.

Instructional strategies from the examples

- Guide children to identify basic repeating patterns in activities and in the classroom.
- Introduce errors into patterns and challenge children to identify and correct the error.
- Build children's ability to extend patterns by having them predict what would come next.

South Carolina standards alignment

MATHEMATICS: PS.7, 1.ATO.9a, 1.ATO.9b
TEACHERS: No direct alignment

First, teachers should ask children to experiment with basic repeating patterns, such as having them select a pattern by which everybody in the class will line up to go to lunch (for example, hot lunch, bag lunch, hot lunch, bag lunch, hot lunch, bag lunch). Once children understand simple patterns, they should be challenged to construct more complex ones (for example, hot lunch, hot lunch, bag lunch, hot lunch, hot lunch, bag lunch, hot lunch, hot lunch, bag lunch). Teachers can then challenge children to notice patterns in the classroom, such as tiles on the floor, stripes on clothing, bricks on a wall, or even seasons in the year. Teachers should also introduce errors to children to further challenge their ability to recognize and detect patterns.

After children demonstrate an understanding of patterns, teachers should ask them to use their understanding to predict what comes next. Teachers can give children a string of beads that follow a pattern and ask them to continue the pattern along the string.

Teachers can introduce complexity by first asking children to create a pattern based on instructions (two blue beads and then two yellow beads) and then proceeding to introduce new colors or characteristics (small or large beads). Lastly, children should design patterns themselves. See Example 5 on page 31 in the practice guide referenced on the last page of this document for a sample activity to create and extend patterns.

3. Promote children's understanding of measurement by teaching them to make direct comparisons and to use both informal or nonstandard (e.g., the child's hand or foot) and formal or standard (e.g., a ruler) units and tools.

Instructional strategies from the examples

- Use simple examples to help children identify differences in measure between objects (e.g., direct comparison of length to determine which is longer).
- Provide opportunities for children to measure with both formal and informal units.

South Carolina standards alignment

MATHEMATICS: K.MDA.2, 1.MDA.2, 2.MDA.1
TEACHERS: INST.MS.2, PLAN.SW.3

Teachers should begin with simple examples to highlight differences in measurement among objects and show how to directly compare them. For example, teachers can present children with two different-sized crayons, ask which one is longer, and then guide them to directly compare the lengths by placing the crayons side by side. Once children demonstrate an understanding of direct comparison, teachers can provide them with tasks such as ordering a set of four to five crayons from shortest to longest. Teachers should reinforce measurement vocabulary when making comparisons.

Examples of vocabulary words for types of measurement

Type of Measurement	Examples of Vocabulary Words
Length	Long, longer, longest; short, shorter, shortest
Size	Small, smaller, smallest; big, bigger, biggest
Temperature	Warm, warmer, warmest; cold, colder, coldest
Time	Early, earlier, earliest; late, later, latest
Weight	Heavy, heavier, heaviest; light, lighter, lightest

Note. Adapted from Table 5 on page 32 of the practice guide referenced on the last page of this document.

Next, teachers can challenge children to measure objects using informal or nonstandard units. For example, children can use their hands to measure the length of a piece of paper and express the measurement in numeric units (the paper is three hands long). After experience with informal units, children should be introduced to standard measurement units (inches, feet, ounces, pounds) and use standard measurement tools (rulers and scales) to measure objects. Starting with nonstandard measurement before moving to standard measurement reinforces that nonstandard measurement may give different results, while standard measurement does not. Teachers can ask multiple children to measure a length (such as the distance between two desks) using both standard and nonstandard units and compare the results. Standard measurement does not change between children, while nonstandard does. Everyday measurements, such as changes in temperature, time, weight, and height, provide opportunities for children to apply and expand their knowledge.

4. Help children collect and organize information, and then teach them to represent that information graphically.

Instructional strategies from the examples

- Use counting and sorting physical objects as a way to help children begin to see visual representations.
- Encourage children to identify and discuss differences between two sorted sets.
- Guide children to use visual representations (e.g., tally marks) to represent the number of objects counted in a set.

South Carolina standards alignment

MATHEMATICS: PS.4, K.MDA.3, K.MDA.4, 1.MDA.4
TEACHERS: INST.MS.2

Teachers should begin familiarizing children with the concept of grouping and visually representing information by asking children to count and sort tangible objects (for example, blocks, crayons) and abstract concepts (for example, 4-year-olds and 5-year-olds). Teachers should challenge children to identify characteristics that lead to grouping objects or individuals (for example, children with different pets, the shape of blocks). Children should recognize all characteristics that differentiate sets (shape, size, color) and then count the number of objects or individuals in each set. The two goals are to identify the characteristics that represent each set and compare the number of items in each set. Next, teachers should ask children to visually represent the information they collect, such as in graphs. Teachers should begin with simple tallies and progress to more complex graphs. For a sample activity for teachers, see Example 6 on page 34 in the practice guide referenced on the last page of this document.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
It is challenging enough to cover everything I need to cover in a day without having to think about four more early math content areas.	Teachers can cover multiple math areas during one lesson. For example, they can ask children to find a collection of objects, count the items, and arrange them in a pattern, all in one lesson. Math games that can be played during transitions or downtime, such as “I spy,” can help teachers find time for math learning.
Some children are struggling with basic vocabulary skills or are being exposed to English for the first time.	Teachers can use visual representations of vocabulary concepts. Or, when multiple children in a classroom speak the same non-English language, teachers can assist English-speaking children in learning to count in their classmates’ native languages. Songs and fingerplays are also helpful for learning math vocabulary. Having children arrange materials or draw to display answers can also help to overcome the language gap.

Reference: Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children* (NCEE 2014-4005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/18

This document provides a summary of Recommendation 3 from the WWC practice guide *Teaching Math to Young Children.* Full reference at the bottom of last page.

CONTENT: **Mathematics**
GRADE LEVEL(S): **preK–K**
LEVEL OF EVIDENCE: **Minimal**

Recommendation

Use progress monitoring to ensure that math instruction builds on what each child knows.

Progress monitoring is a helpful method for making sure that math instruction is deliberate and useful for children. Teachers can monitor progress to tailor lessons and instruction based on children's current math skill levels. This method helps to ensure that children are receiving math instruction that is difficult enough so that they are always learning.

How to carry out the recommendation

1. Use introductory activities, observations, and assessments to determine each child's existing math knowledge, or the level of understanding or skill they have reached on a developmental progression.

Instructional strategies from the examples

- Use introductory activities to determine children's ability in working with a new concept.
- Carefully observe children during a math activity, asking questions that require children to think out loud and describe their problem-solving process.
- Use formal assessments to guide planning and instruction.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: INST.MS.2, PLAN.IP.3, PLAN.SW.3

To begin progress monitoring, teachers should find each child's level of skill and knowledge in math through *introductory activities*, observations, and formal assessments. Teachers can use introductory activities to present new concepts and determine what children can complete independently. For example, after instructing children on shapes, teachers can include an activity in which the children cut out shapes such as circles and triangles from magazines. Teachers can then ask the children to discuss the shapes and their different sizes. Such activity allows teachers to see whether the children know shapes and can talk about them.

Observations include addressing specific math competencies through activities and watching children in their process of completing the activities. While children are engaged in the activities, teachers can ask questions to see if they can verbalize their process for completing the activity. If children can complete the activity correctly and explain their process verbally, they can move on in the lesson.

Teachers can plan instruction and lessons by looking at children's overall performance on *formal assessments* to identify their skill levels. Teachers can also look at children's answers to specific questions or in specific sections of assessments for a deeper understanding of their knowledge and skills. This information can be valuable in selecting appropriate goals for instruction.

2. Tailor instruction to each child's needs, and relate new ideas to their existing knowledge.

Instructional strategies from the examples

- Use information collected from Step 1 above to guide instruction.
- Build activities based on the next level of development and connect them to children's interests.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.2a
TEACHERS: PLAN.IP.3, PLAN.SW.3

Teachers should use learnings from introductory activities, observations, and formal assessments to identify where children's knowledge and skills place them in the developmental progression. Doing so will help teachers determine the next steps in the learning process and create instructional activities aligned to the next levels. For example, when children demonstrate they can use subitizing to determine which set of objects has more (for example, a set of four has more than a set of three), they can use meaning counting to determine which collection contains more. Once children can count a set of 10 objects, teachers can include a set of 11 or more objects to increase the difficulty. See Table 3 on page 13 of the practice guide referenced on the last page of this document for a sample developmental progression related to number knowledge.

Teachers should not only create activities aligned to the next level in the learning progression but also connect new knowledge to children's interests. For example, if children like art, teachers can create activities that include drawing, such as having children draw eight dogs and 10 dogs and then describe which set has the most dogs.

Children might be at different levels in a learning progression, so it's helpful to group them by level for some activities. For example, some children might be able to count sets of 10 objects, whereas others might be able to count sets of eight. Teachers can group children who can count to 10 and group those who can count to eight. Then, they can observe the groups and increase the difficulty of the activity as the child's ability levels rise.

3. Assess, record, and monitor each child's progress so that instructional goals and methods can be adjusted as needed.

Instructional strategies from the examples

- To understand children's learning, use repeated cycles of implementation, assessment, and planning.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: PLAN.IP.3, PLAN.SW.3, PLAN.A.2

Teachers can use progress monitoring to assess children's progress through introductory activities, observations, and formal assessments. Progress monitoring involves first selecting an activity using a developmental progression, then repeatedly cycling through a process of implementing the activity, assessing child's levels (using the developmental progression), and planning or selecting additional activities. Teachers might use a checklist such as the one below to monitor progress during a sample activity.

Progress monitoring checklist

Activity: Which set has the most triangles?	Child	Date	Counted Correctly?	Decided Correctly?	Errors Made
Set 1: ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲	Suzy	November	Yes	No	Selected set 1 as having more triangles
Set 2: ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲	Billy	November	No	Yes	Counted eight twice for set 2

Note. Adapted from Example 8 on page 40 of the practice guide.

When assessments show that children's math knowledge and skills are growing, teachers can plan activities, following the developmental progression, that are increasingly more difficult and continually assess children's math levels as they complete the new activities.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
I already use solved problems during whole-class instruction, but I'm not sure children are fully engaged with them.	Ask questions, and be sure to include all children in the discussion to motivate them to think critically. Model think-aloud questions (for example, "Will the strategy work for every problem like this?" "Why or why not?" "How would you modify the solution, if you can, to make it clearer to other children?"). See Examples 1.1 and 1.2 in the practice guide. Additionally, use solved problems beyond whole-group settings to be sure they are scrutinized in more meaningful ways. Include solved problems in class assessments to make whole-class work relevant to children. See Examples 1.9, 1.10, and 1.11 in the practice guide.
I do not know where to find solved problems to use in my classroom and do not have time to make new examples for my lessons.	Find sample or worked problems in published curricular materials. Use past or current de-identified child work (such as homework, projects, and assessments) as other examples, particularly for unique solution paths or incorrectly solved problems. Share across classrooms to increase your access.
I'm worried that showing children incorrectly solved problems will confuse them.	Although children may not be familiar with examining incorrectly worked problems, doing so can help them build important critical-thinking skills. Be sure that children are clearly aware that a problem contains an error, then focus on the steps to understand the process and where it went wrong. Fully discuss each step to prevent confusion and build recognition and understanding of how the error occurred. See Examples 1.5, 1.6, and 1.7 in the practice guide.

Reference: Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children* (NCEE 2014-4005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/18

This document provides a summary of Recommendation 4 from the WWC practice guide *Teaching Math to Young Children.* Full reference at the bottom of last page.

CONTENT: **Mathematics**

GRADE LEVEL(S): **preK–K**

LEVEL OF EVIDENCE: **Minimal**

Recommendation

Teach children to view and describe their world mathematically.

Teachers can start by having children describe math they see and experience in the real world informally, using their own language. Once children are comfortable with informal representations and talking about math, they can move to more formal representations and vocabulary, with teacher support. Applying math to real-world situations while children play and work at school will help them develop a better understanding of the math concepts they are learning.

How to carry out the recommendation

1. Encourage children to use informal methods to represent math concepts, processes, and solutions.

Instructional strategies from the examples

- Introduce new math concepts using terms and connections familiar to children.
- Focus on having children explain their mathematical thinking and reasoning in a way that makes sense to them, rather than initially focusing on correct mathematical language.

South Carolina standards alignment

MATHEMATICS: K.G.4, 1.MDA.2
TEACHERS: INST.MS.2

Teachers should begin math instruction by informally representing math concepts through connections to experiences, using recognizable terms and comparisons. For example, teachers should think about the vocabulary that children already know when teaching them addition. Children are likely familiar with the terms “take away” and “left.” Instead of using math terms for subtraction, teachers might say, “Suzy had four candy bars, and her mom took away one candy bar. How many candy bars does Suzy have left?"

Using information representations of math concepts

Concept	Informal Representation	Teaching the Concept
Whole number	"three"	Collections of blocks, dots, tally marks, fingers, or other countable objects can represent numerals. For example, when playing a game, use blocks to represent children's scores so that everyone can track each player's score.
Equal	"same number as" or "same as"	Provide opportunities for children to begin to recognize that collections that have the same number when counted are equal. For example, a collection of four plates is the same number as a collection of four cups.
Unequal	"more than" or "fewer than"	Point out that a collection is more (or fewer) than another if it requires a longer (or shorter) count. For example, seven is more than six because it requires counting beyond six.
Addition	"and" or "more"	Start with a collection and add more items to make it larger. For example, start with three crayons and add one more. Then ask, "How many?"
Subtraction	"take away" or "fewer"	Start with a collection and take away some items to make it smaller. For example, start with three crayons and take away one. Then ask, "How many?"

Note. Adapted from Table 6 on page 43 of the practice guide.

2. Help children link formal math vocabulary, symbols, and procedures to their informal knowledge or experiences.

Instructional strategies from the examples

- Help children connect their informal language to formal mathematical language through modeling.
- Use formal mathematical terms throughout the school day.
- Connect children's mathematical language to mathematical symbols.

South Carolina standards alignment

MATHEMATICS: PS.1a, PS.4
TEACHERS: PLAN.SW.3

Once children are comfortable with talking about and representing math concepts informally, teachers can begin introducing formal math concepts. Teachers can do so by connecting informal representations with formal math terms (for example, "more" with "addition"). To help children learn formal terms, teachers should be sure to use each term multiple times throughout the school day and also provide opportunities for children to repeatedly use the term. For example, when teachers read a book with the class, they can make connections to math terms by talking about the characters in the book, such as one character being younger or older than another.

Children must also learn to connect their informal math knowledge with formal math symbols (for example, "more" connects with +). An example activity might involve having children solve addition or subtraction problems with objects in the classroom. For more ways to connect informal knowledge to formal math symbols, see Table 7 on page 44 of the practice guide referenced on the last page of this document.

3. Use open-ended questions to prompt children to apply their math knowledge.

Instructional strategies from the examples

- Use open-ended questions to help children build their ability to reason mathematically and use math vocabulary.

South Carolina standards alignment

MATHEMATICS: PS.1, PS.2, PS.3
TEACHERS: INST.MS.2

Teachers can use open-ended questions ("what," "why," "how," and so on) to prompt children to think about math concepts and vocabulary. These questions allow children to apply what they know in answering a question. Open-ended questions should encourage children to use concepts and terms they are familiar with. For example, teachers might show a picture of two trees and ask, "What makes these two trees different?" This question invites multiple answers. One child might say, "The first tree is taller than the second tree." Another child might say, "They are different kinds of trees." Allowing for multiple responses lets children discuss concrete objects using math terms. Sample open-ended questions that teachers might use to encourage children to discuss math concepts are provided below. See the practice guide referenced on the last page of this document for more examples.

Examples of open-ended questions

- What makes these the same/different?
- How can you tell how ___ (tall, short, long, wide) that is?
- How did you figure out how many ___ (blocks, buttons, shapes) there are?
- Why do you think this is ___ (taller, shorter, longer, wider) than that?

Note. Adapted from Table 8 on page 45 of the practice guide.

4. Encourage children to recognize and talk about math in everyday situations.

Instructional strategies from the examples

- Provide opportunities for children to discuss connections between math concepts and the world around them.
- Have children explain their solution strategies out loud.
- Encourage children to think about other ways they might be able to solve a problem.

South Carolina standards alignment

MATHEMATICS: PS.2a
TEACHERS: INST.MS.2

Teachers can strengthen children's math knowledge by providing opportunities for them to connect math concepts to the real world. For example, teachers might ask children for help in deciding the number of pencils needed for the classroom: "How should we figure out how many pencils we need for the classroom?" Teachers could then encourage the children to think about the solution path for the problem by discussing it out loud: "How did you think of that answer? What steps did you take?" Once the children have explained how they solved the problem, teachers can repeat the explanation out loud and then ask, "Are there different steps we could have taken to solve this problem?" If children are struggling to discuss their mathematical process, teachers can provide examples. This activity allows children to connect their math knowledge to problem-solving in everyday situations.

Potential roadblocks and how to address them

Roadblock	**Suggested Approach**
I'm not sure what types of open-ended questions are most effective for getting young children to think mathematically.	Use the strategies described in this recommendation to think about open-ended questions you might use. Start a math conversation by asking, "How can we figure this out?" Once children have answered, prompt them to think about how they arrived at their answers by asking, "How did you think of that answer?" and "What steps did you take?" If there is more than one correct strategy, ask children, "Are there other steps we could have taken to solve the problem?"

Reference: Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children* (NCEE 2014-4005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/18

This document provides a summary of Recommendation 5 from the WWC practice guide *Teaching Math to Young Children.* Full reference at the bottom of last page.

CONTENT: ***Mathematics***

GRADE LEVEL(S): ***preK–K***

LEVEL OF EVIDENCE: ***Minimal***

Recommendation

Dedicate time each day to teaching math, and integrate math instruction throughout the school day.

Each school day should have time set aside for math. Teachers should connect math concepts to familiar objects so children can see the relationship to everyday situations. It is also helpful to incorporate math concepts into lessons for other subjects so children begin to understand how those subjects are connected to math. Including math-related items, such as games, in the classroom can help children apply the math concepts they have learned in new contexts.

How to carry out the recommendation

1. Plan daily instruction targeting specific math concepts and skills.

Instructional strategies from the examples

- Explicitly plan time during each day for children to develop math skills.
- Provide opportunities throughout the school day for children to apply their math concepts.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: PLAN.IP.3, PLAN.A.2

Children in the lower grades should have explicit time for developing math skills during each school day. During this time, children can learn specific math concepts through whole- and small-group activities. Then, teachers can provide opportunities throughout the remainder of the day for children to apply the math concepts they have learned. Children may be at different math levels, so teachers should consider how best to group them for each activity. For example, teachers might introduce a new math concept as a whole group and then divide children into smaller groups by math level to strengthen the learning through an activity or game.

Linking large groups to small groups

Objective	Understand the differences and similarities among triangles, rectangles, and squares.

Materials needed	• Book: *Bear in a Square*, by Stella Blackstone
	• A variety of other objects (based on availability, but could include the following):
	  ○ Large pieces of paper cut into various shapes for painting
	  ○ Lunch trays and a small amount of sand
	  ○ Geoboards with rubber bands

Directions: large group	Read the book in a large group, highlighting the names of all the shapes but focusing specifically on the difference between the number and length of sides and types of angles in triangles, rectangles, and squares.
Directions: small group	Once children are divided into small groups, highlight the number and length of sides and types of angles in each of the shapes the children create during activities like the one below. Children should be encouraged to use informal terms to describe the shapes. Provide paint, chalk, or other art materials so that children can add a stripe around the edge of a large paper cutout of a triangle or rectangle. Then, have the children continue to add more of the same shapes inside the original shape to create a design with concentric shapes.
Early math content areas covered	Geometry (shapes and attributes of shapes)
Integrating the activity into other parts of the day	Take a group walk outside to collect sticks of different sizes, and then use them to make and identify shapes.
Using the activity to increase math talk in the classroom	When children locate a shape in the classroom environment, ask them to explain it to the group: "How can you tell that shape is a ___?" Prompt the children to identify the number and length of sides and type of angles.

Note. Adapted from Example 9 on page 49 of the practice guide.

2. Embed math in classroom routines and activities.

Instructional strategies from the examples

- Look for ways to embed math into everyday classroom routines and activities.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: INST.MS.2

Teachers can use opportunities within classroom routines and activities to allow children to practice the math concepts they have learned during math lessons. For example, teachers can have children count the number of children in attendance at snack time and then count out enough apples for each child to have one. Teachers could extend this activity and have the children keep track of the counts over two different days, then make comparisons between the two days. See Example 10 on page 50 of the practice guide referenced on the last page of this document for more detail about how math might be integrated into snack time.

3. Highlight math within topics across the curriculum.

Instructional strategies from the examples

- Look for ways to integrate math into other content areas as it makes sense.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: INST.MS.2, PLAN.SW.3

Teachers can provide children with opportunities to apply math concepts in other content-area lessons. Depending on where children are in their learning process, teachers can include time for counting, looking at shapes, or using other math concepts during instruction in other content areas.

Examples of how a teacher can incorporate math into other content-area lessons

	Math Content Area				
	Number and Operations	Geometry	Patterns	Measurement	Data Analysis
Literacy	*We All Went on Safari*, Krebs	*Bear in a Square*, Blackstone	*A Pair of Socks*, Murphy	*How Big Is a Foot?* Myller	*It's Probably Penny*, Leedy
Science	Count collections of natural objects.	Describe objects from nature in geometric terms.	Find and identify patterns in nature.	Measure the growth of a plant in the classroom each day.	Graph the amount the classroom plant grows each day.
Art	Count how many objects appear in a piece of artwork.	Identify shapes in artwork.	Use patterns to make pictures or frames for pictures.	Measure to make frames for art out of poster board or card stock.	Make a graph of the children's favorite colors.
Health and Safety	Count the length of time it takes to wash your hands.	Use traffic signs to recognize shapes.	Jump rope or play hopscotch with an alternating pattern.	Measure your body's growth over time.	Graph your height or foot size.

	Math Content Area				
	Number and Operations	**Geometry**	**Patterns**	**Measurement**	**Data Analysis**
Social Studies	In a unit about families, order people by size or from youngest to oldest.	Identify squares, straight lines, curved lines, and other shapes on maps.	Study patterns in clothes from different parts of the world.	Make a map of the neighborhood, using measuring, geometry, spatial thinking, and positioning words.	Graph the size of the children's families.

Note. Adapted from Table 9 on pages 51–52 of the practice guide.

4. Create a math-rich environment where children can recognize and meaningfully apply math.

Instructional strategies from the examples

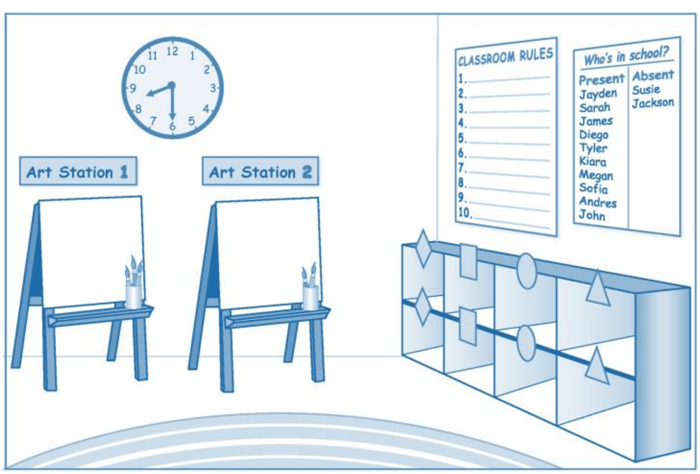
- Provide accessible objects and tools throughout the classroom that are related to math concepts learned in the class.
- Encourage children to think about how to apply math concepts to everyday activities.
- Engage children in labeling tools and materials that can be used for math activities.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: INST.MS.2

Teachers can create a classroom environment that is well-supplied for children to apply math. As an option, teachers can have accessible objects and tools in the classroom that are related to learned math concepts. Teachers might also have children apply math concepts to everyday activities and tools that are already part of the classroom routine, including number lists, blocks, beads, rulers, sorting bins, and so on. Children may not directly know how to use the tools to think about the math concepts they have learned, so teachers might model how to use them during instruction and group work. The tools should be labeled and placed in locations accessible to children. Teachers should also consider including children in the labeling process so they understand what each label means. Labeling is also an excellent opportunity for children to apply math concepts such as counting.

Example of a math-rich environment in the classroom



Note. Taken from Figure 7 on page 53 of the practice guide.

5. Use games to teach math concepts and skills, and to give children practice in applying them.

Instructional strategies from the examples

- Use games to engage children in math.
- Align games with the math concepts children are learning in class and their level of understanding of these concepts.

South Carolina standards alignment

MATHEMATICS: No direct alignment
TEACHERS: No direct alignment

Teachers can use games to immerse children in math concepts. Games might motivate children to engage in math in a fun way. Games should align with the math concepts being learned in class and with the children's levels. Curricula will often include games, but individual games can also be purchased elsewhere. Additionally, existing games can be used to apply math concepts. An example of a math game is provided below.

The *Animal Spots* game

Objective	Practice one-to-one correspondence and cardinality.
Materials needed	• Pictures of small animals or materials children can use to draw their own animals • Small circles of paper to use as spots • Glue • A die or spinner to determine the number of spots to place on each animal
Directions	Have each child draw the outline of an animal on a piece of paper, or provide handouts with large outlines of animals. Each child should take a turn throwing the die to determine how many spots to place on their animal. The children should count out the number of dots on the face of the die, and then they should choose the same number of "spots" from a bowl of paper circles in the center of the table. After children have selected the correct number of spots, they can glue them onto their animals. Teachers can tailor the *Animal Spots* game for use with the entire class, a small group, or individual children.
Early math content areas covered	• Counting using one-to-one correspondence • Cardinality

Monitoring children's progress and tailoring the activity appropriately	• Observe the play, noting each child's ability to count the number of dots on the die and count out the same number of spots from a larger pile. • Use one die or a spinner at the beginning; then, use two dice to increase difficulty.
Integrating the activity into other parts of the day	Have children count out objects from a larger set. For example, a child can choose 10 blocks for building or five shapes from a larger collection to use for a collage.

Note. Adapted from Example 11 on page 54 of the practice guide.

Potential roadblocks and how to address them

Roadblock	Suggested Approach
The school is on a limited budget and cannot afford to purchase many classroom materials or games.	To enhance instruction without having to spend money, teachers can use existing opportunities throughout the day, objects already in the school environment, and tools that others have created, such as those found online. When teachers or schools decide to purchase items, they should be strategic by thinking about what will best engage children in their current learning.
I am told that it is important to provide literacy-, science-, art-, and math-rich environments. It is difficult to keep all subjects in mind at all times.	When planning lessons, think about how to incorporate aspects of various content areas into each lesson. All content areas do not need to be incorporated at one time; math can be included in different subjects at different times. Be strategic about how to meet the learning goals of the day across content areas. See Table 9 on pages 51–52 of the practice guide referenced on the last page of this document for ideas.
I do not have my own space because multiple classes use the same classrooms throughout the day.	If teachers cannot easily change the classroom, they can consider how to use mobile objects to enhance the environment, such as transportable games or charts.
Parents may wonder why their children are playing games in school.	Teachers can be strategic about choosing games to enhance math instruction and learning. They should connect games to specific learning objectives. Doing so will help parents see the connections between the games and instruction and learning. Additionally, teachers should explain how important play is in the development of young learners.

Reference: Frye, D., Baroody, A. J., Burchinal, M., Carver, S. M., Jordan, N. C., & McDowell, J. (2013). *Teaching math to young children* (NCEE 2014-4005). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/18